

Previous Lecture

- ◆ Initial Value Problem
- ◆ General/Particular Sols
- ◆ Family of Sols



1.3: Slope Fields and Solution Curves

How to solve: $\frac{dy}{dx} = f(x, y)$ (w/y on the RHS)? (e.g., $y' = x^2 y dx$) Like this? $y = \int f(x, y) dx$??? (e.g., $\int x^2 y dx$)



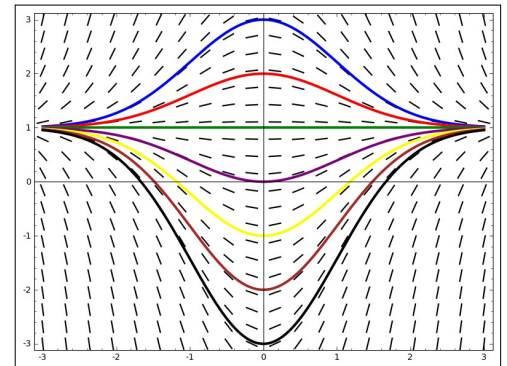
In fact, not all DEQs are solvable!



Since $f(x, y)$ may involve the unknown function $y(x)$, integrating $f(x, y)$ with respect to x may not be possible.
We need another method!

When all else fails, we can draw a **direction field** (or **slope field**) for $y' = f(x, y)$.

- ◆ Choose any point (x_0, y_0)
- ◆ Plug it into $f(x_0, y_0)$, this gives you a slope (a number).
- ◆ Graph a short line at (x_0, y_0) having slope $y' = f(x_0, y_0)$.
- ◆ Repeat as needed.



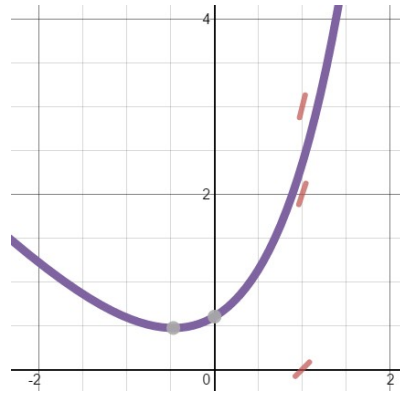
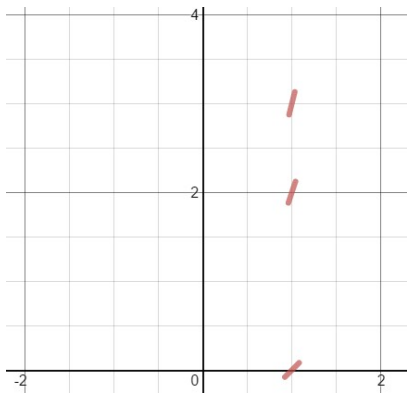
Slope marks with particular sols

Drawing paths parallel to nearby slope marks gives **solution curve approximations**, representing particular sols to the DEQ.

Example: $y' = x + y$ w/ initial value $y(0) = 1$.

Determine slope at $(1, 2)$. By substitution: $y' = 1 + 2 = 3$.

Similarly, slope at $(1, 3)$ is 4, and slope at $(1, 0)$ is 1 ...



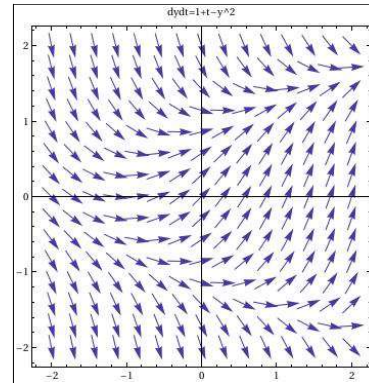
Geogebra Classic 6: [geogebra.org/download](https://www.geogebra.org/download)

Click "All download options," then download "Direction Fields GeoGebra File" from Canvas.

WolframAlpha.com

Type your DEQs into wolfram. If $y' = f(x, y)$, then use syntax like: "slope field $f(x, y)$ "

For $\frac{dy}{dx} = 1 + x - y^2$, this would be "slope field $1 + x - y^2$ ".



Existence and Uniqueness of Solutions

No one wants to search for something that doesn't **exist**!

Do DEQs $y' = f(x, y)$ have solutions y for all initial conditions?

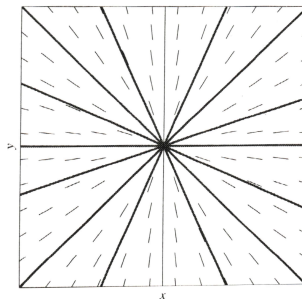
Example: $y' = \frac{1}{x}$, $y(0) = 1$.

$$\int \frac{1}{x} dx = \ln|x| + c.$$

Solution not defined for initial condition $y(0) = 1$, (since $\ln x$ isn't defined when $x = 0$).

If we *DO* have a solution to a DEQ, we'd also like to be sure there aren't any other solutions going through our initial condition.

That is, we want the solution to be **unique**.

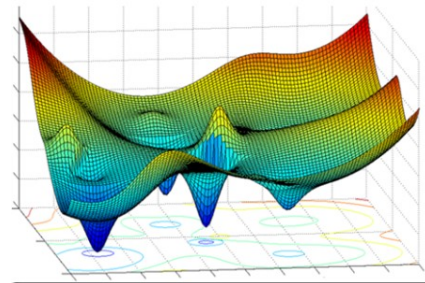


e.g., sols to DEQ: $xy' = y$ that pass thru the origin aren't unique

Existence and Uniqueness Criteria

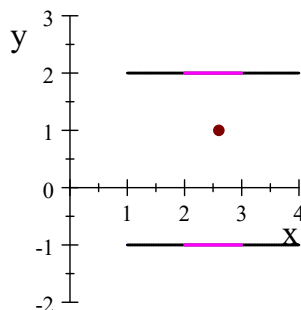
Let: $\frac{dy}{dx} = f(x,y)$, with initial cond. $y(a) = b$.

Assume both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ (derivative w.r.t. the dependent var) are continuous on some rectangle R (containing (a,b) in its interior).



continuous function of two variables x,y

[Let's say f and $\frac{\partial f}{\partial y}$ are cont. in the rectangle $x \in (1,4)$ and $y \in (-1,2)$ where $(a,b) = (2.6, 1.0)$.]



f and $\frac{\partial}{\partial y}f$ is continuous in this rectangle R .

Then, **Theorem 1.3** from the book guarantees the existence & uniqueness of a **local** solution $y(x)$ in some interval $x \in I$ (still containing a) within R , but possibly smaller than the width of R .

[Continuing our previous example, the interval I (pink in the graph above) might turn out to be $x \in (2,3)$. Observe $a = 2.6$ is still in this interval.]

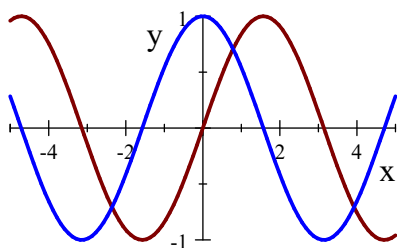
Particularly, continuity of $f(x,y)$ guarantees existence on some I .

And continuity of $\frac{\partial}{\partial y}f(x,y)$ guarantees uniqueness of that solution.

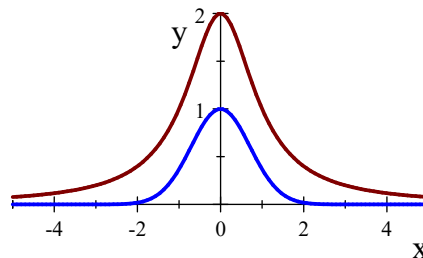
In addition to being continuous, if $\frac{\partial}{\partial y}f(x,y)$ is also **bounded** for all x and y , then **global** existence/uniqueness (on $\mathbb{R}$) of a solution y is guaranteed.

But which functions are bounded?

Polynomials are not ($y \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$), but the following functions are: $\sin x$, $\cos x$, $\frac{2}{1+x^2}$, $e^{-|x|^2}$, etc.



$\sin x, \cos x$



$\frac{2}{1+x^2}, e^{-|x|^2}$

Another way to ensure that you have global existence (on $\mathbb{R}$) is to check if the DEQ is **linear first-order** (i.e. $ay' + by + c = d$). **ALL** such DEQs have global existence for their solutions.

Linear means that each instance of $y, y', y'', \dots$ occurs by itself in each term, and to the first power.

So, $ay' + by + c = d$ is linear, but $(y')^2 = 4$ is not. Nor is $ay' + b\sqrt{y} + c = d$ nor $ay' + b\frac{1}{y} + c = d$ nor $ay'y = d$.



Negative results from the above tests **tell you nothing**. In other words, even if these tests fail,

the solution may still have existence/uniqueness/globalness! (you may have to think about this statement to get it.)

Example: Determine if $xy' = 2y$ with $y(-1) = 1$ has a unique solution.

$$y' = \frac{2}{x}y$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}, \text{ when } x, y(x) \neq 0 \text{ (ok near our init. cond.)}$$

$$\int \frac{1}{y} dy = \int \frac{2}{x} dx \Rightarrow \ln|y| = 2\ln|x| + c = \ln|x^2| + c \Rightarrow$$

$$y = \pm e^c x^2 = Cx^2, \text{ where } C \in \mathbb{R}. \quad (\text{I included zero since } y(x) \equiv 0 \text{ is a solution})$$

Notice that $f(x, y) = \frac{2}{x}y$ and $\partial_y f = \frac{2}{x}$ are discontinuous at $x = 0$.

We'll see that this causes uniqueness to be lost.

What is C when $y(-1) = 1$?

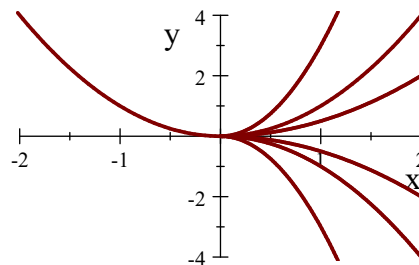
So, $C = 1$.

It turns out that the above discontinuity results in our ability to choose a *different arbitrary constant* on the other side of $x = 0$.

Let's call the constant C_2 to the right of $x = 0$.

Now, observe that $y(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ C_2 x^2 & \text{if } x > 0, \end{cases}$ is a continuous solution for *any* $C_2 \in \mathbb{R}$.

(substitute them into $y' = \frac{2}{x}y$ to check if they are sols, and substitute $x = 0$ in to check for continuity)



Sols for $y' = \frac{2}{x}y, y(-1) = 1$.

So, uniqueness was lost.

Exercises 1.3

What did we learn?

- ◆ Direction/Slope Fields of DEQs
- ◆ Existence and Uniqueness of Sols Criteria



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Materials for Other Courses Found at **MathTalker.org**