

Calculus I

5.5 - Substitution Rule

Substitution Rule:

Recall that due to the chain rule we have: $(\sin(e^{2x}))' = \cos(e^{2x})(2e^{2x})$, or $(f(g))' = F(g) \cdot g'$. And since integrating is an opposite process from differentiating, we should be able to determine that $\int F(g) \cdot g' = f(g)$.

So, let's say you had to integrate something of the form $\int F[g(x)] \cdot g'(x)dx$, or notated differently:

$\int F(u)du$, when $u = g(x)$. For example $\int \cos(e^{2x}) \cdot 2e^{2x}dx$. How could we go about solving this?

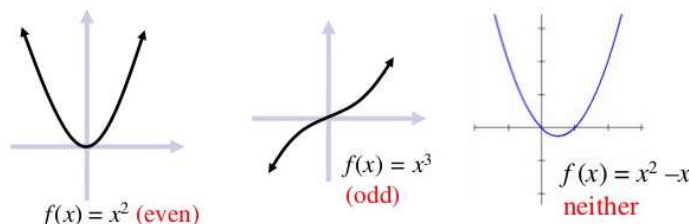
Notice that the expression $\int F(u)du$ is simpler than $\int F[g(x)] \cdot g'(x)dx$. But how did we eliminate $g'(x)$ and switch from dx to du ? Until now, we have treated the dx as a notational appendage which merely told us which variable we were integrating over. The substitution rule works because it uses dx operationally. Notice that if we take the derivative of $u = g(x)$ with respect to x , we get $\frac{du}{dx} = g'(x)$, or $du = g'(x)dx$. This allows us to make the necessary substitution above (try it!).

Substitution with Definite Integrals: $\int_{x=a}^{x=b} F(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} F(u)du$,
notice you have to change the bounds of integration!

Integrals of Symmetric Functions: If F is continuous on $[-a, a]$, and...

♦ If F is even, then $\int_{-a}^a F(x)dx = 2 \int_0^a F(x)dx$.

♦ If F is odd, then $\int_{-a}^a F(x)dx = 0$.



Useful Trigonometric Integral: $\int \tan x dx = \ln|\sec x| + C$.

Problem 46. Evaluate the Indefinite Integral: $\int x^2 \sqrt{2+x} dx$

Let $u = 2 + x$. Then $\frac{du}{dx} = 1$ or $du = dx$,

$$x = u - 2, \text{ so } x^2 = (u - 2)^2.$$

$$\text{Therefore: } \int x^2 \sqrt{2+x} dx = \int (u - 2)^2 \sqrt{u} du$$

$$= \int (u^2 - 4u + 4)u^{\frac{1}{2}} du = \int \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right) du$$

$$= \frac{2}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + C.$$

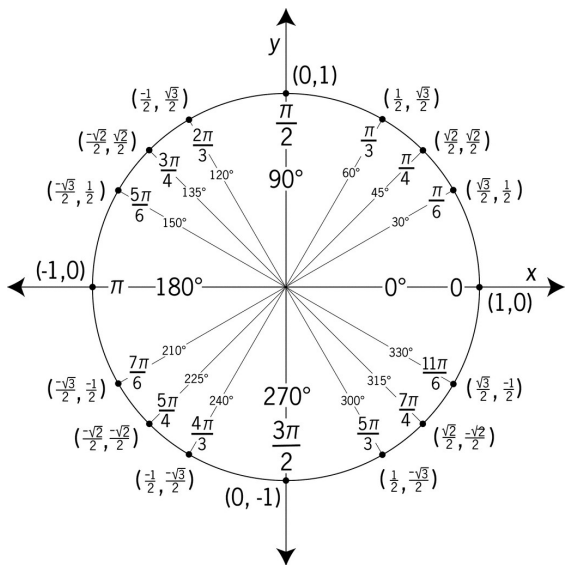
Are we done?

$$= \frac{2}{7}(2+x)^{\frac{7}{2}} - \frac{8}{5}(2+x)^{\frac{5}{2}} + \frac{8}{3}(2+x)^{\frac{3}{2}} + C.$$

Problem 70. Evaluate the definite integral: $\int_{\frac{1}{2}}^1 \frac{\cos^{-1}x}{-\sqrt{1-x^2}} dx.$

Notice the form $\frac{1}{-\sqrt{1-x^2}}$ is related to $(\cos^{-1}x)'$.

So, let: $u = \cos^{-1}x$, so $du = \frac{dx}{-\sqrt{1-x^2}}.$ What else do I need to change?



When $x = \frac{1}{2}$, $u = \pm \frac{\pi}{3}$; when $x = 1$, $u = 0$.

$$\text{Thus, } \int_{x=\frac{1}{2}}^{x=1} \frac{\cos^{-1}x}{-\sqrt{1-x^2}} dx = \int_{u=\pm \frac{\pi}{3}}^{u=0} u \, du$$

$$= \left[\frac{u^2}{2} \right]_{\pm \frac{\pi}{3}}^0 = \left(\frac{(0)^2}{2} - \frac{\left(\pm \frac{\pi}{3}\right)^2}{2} \right) = -\frac{\pi^2}{18}.$$

What didn't I need to change back to x this time, and why?

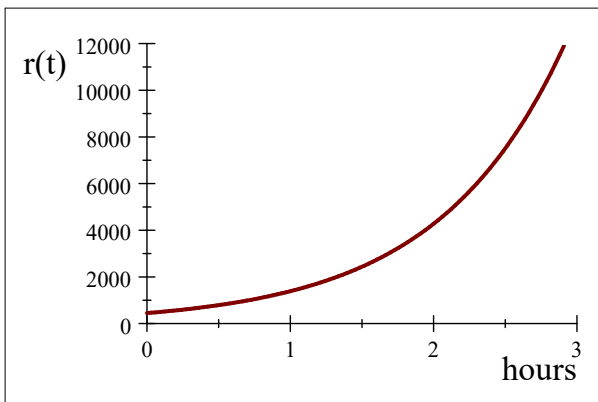
Instead of changing the bounds of integration, we have the option of simply converting back to x before applying the bounds:

$$\int_{x=\frac{1}{2}}^{x=1} \frac{\cos^{-1}x}{-\sqrt{1-x^2}} dx = \int_{x=\frac{1}{2}}^{x=1} u \, du = \left[\frac{u^2}{2} \right]_{x=\frac{1}{2}}^{x=1} = \left[\frac{(\cos^{-1}x)^2}{2} \right]_{x=\frac{1}{2}}^{x=1} = \left(\frac{(0)^2}{2} - \frac{\left(\pm \frac{\pi}{3}\right)^2}{2} \right) = -\frac{\pi^2}{18}.$$

Either way, we must still consult the unit circle!

Problem 82. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour.

How many bacteria will there be after three hours?



So the growth rate is $r(t) = ae^{bt}$, with $a = 450.268$ and $b = 1.12567$.

Let $P(t) :=$ population after t hours.

Since $r(t) = P'(t) \dots$

$\int_0^3 r(t) dt = P(3) - P(0)$ is the total change in the population after 3 hours (net change theorem).

Since we start with 400 bacteria, the population at hour three will be...

$$P(3) = 400 + \int_0^3 r(t) dt = 400 + \int_0^3 ae^{bt} dt$$

$$= 400 + a \int_0^3 e^{bt} dt$$

$$= 400 + a \left[\frac{1}{b} e^{bt} \right]_0^3 = 400 + \frac{a}{b} (e^{3b} - e^{0 \cdot b}) = 400 + \frac{a}{b} (e^{3b} - 1),$$

and substituting back in our constants:

$$P(3) = 400 + \frac{450.268}{1.12567} (e^{3(1.12567)} - 1) \approx 11,713 \text{ bacteria !?!}$$

