

Calculus I

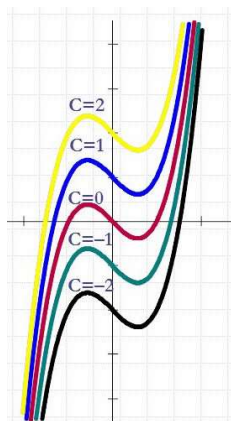
5.4 - Indefinite Integrals and the Net Change Theorem

Indefinite Integrals: $\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

For example $\int x^3 dx = \frac{x^4}{4} + C$, because $\frac{d}{dx} \left(\frac{x^4}{4} + C \right) = x^3$ (for any C !)

Therefore, an indefinite integral $\int f$ represents a **family of functions**,
as opposed to a definite integral $\int_a^b f$ which just represents a *constant*.

And by FTC2: $\int_a^b f(x)dx = \left[\int f(x)dx \right]_a^b = F(b) - F(a)$.



family of functions: $\int f$

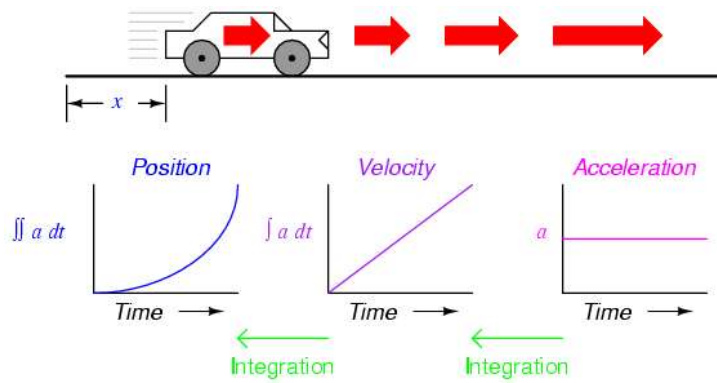
Table of Indefinite Integrals and Properties

$\int cf(x)dx = c \int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int adx = ax + C$, where a is some constant	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int e^x dx = e^x + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (when $n \neq -1$) (backwards power rule)
$\int x^{-1} dx = \int \frac{1}{x} dx = \ln x + C$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \sec^2 x dx = \tan x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

Net Change Theorem

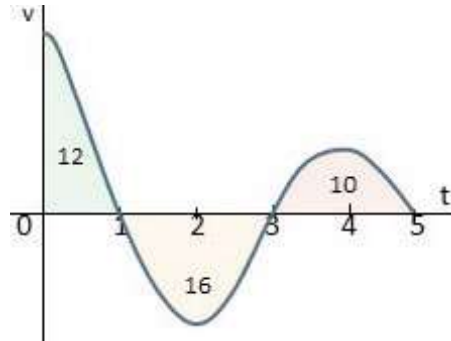
The integral of a rate of change (of some quantity) is the net change (of that quantity) over the interval defined by the bounds of integration. So, $\int_a^b F'(x)dx = F(b) - F(a)$ represents the net change of F over the interval $[a, b]$.

Examples



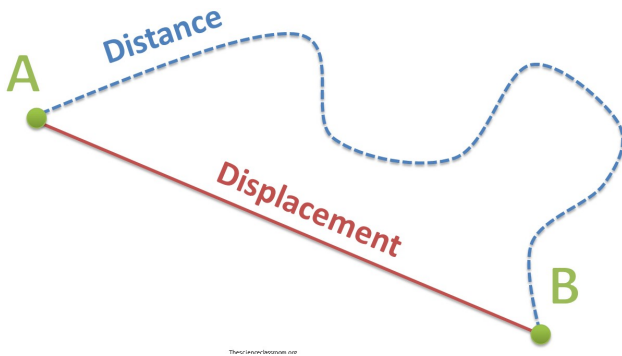
Position $x(t)$ and Velocity $v(t)$ functions:

Displacement of a car's position between t_1 and t_2 (distance from starting position) is calculated by integrating the velocity: $\int_{t_1}^{t_2} v(t) dt = x(t_2) - x(t_1)$



Net change is $12 - 16 + 10 = 6$.

Distance traveled (as recorded by the odometer) between t_1 and t_2 is calculated by integrating the absolute value of the velocity: $\int_{t_1}^{t_2} |v(t)| dt$



Distance traveled

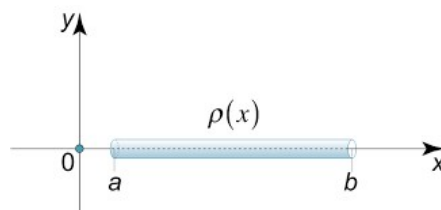
Mass of a Rod:

Notate the total mass of a rod between 0 and x as $m(x)$.

The density can then be calculated as $\rho(x) := m'(x)$

($m'(x)$ is how quickly the total mass is changing as you move along x).

Net change of mass between a and b is then: $\int_a^b \rho(x) dx = m(b) - m(a)$.



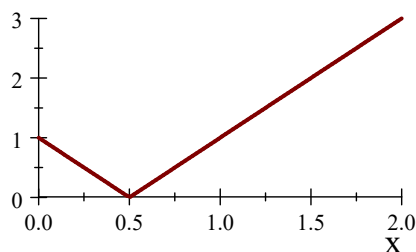
$$|2x - 1| = \begin{cases} 2x - 1 & \text{if } 2x - 1 \geq 0, \\ -(2x - 1) & \text{if } 2x - 1 < 0, \end{cases}$$

$$= \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2}, \\ 1 - 2x & \text{if } x < \frac{1}{2}. \end{cases}$$

$$\text{Therefore, } \int_0^2 |2x - 1| dx = \int_0^{\frac{1}{2}} (1 - 2x) dx + \int_{\frac{1}{2}}^2 (2x - 1) dx$$

$$= [x - x^2]_0^{\frac{1}{2}} + [x^2 - x]_{\frac{1}{2}}^2$$

$$= \left[\left(\frac{1}{2} - \frac{1}{4} \right) - 0 \right] + \left[(4 - 2) - \left(\frac{1}{4} - \frac{1}{2} \right) \right] = \frac{1}{4} + 2 - \left(-\frac{1}{4} \right) = \frac{5}{2}.$$



$$|2x - 1|$$

Problem 54 A honeybee population $n(t)$ starts with $n(0) = 100$ bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t) dt$ represent?

By the Net Change Theorem,

$$\int_0^{15} n'(t) dt = n(15) - n(0) = n(15) - 100, \text{ represents the increase in the bee population over 15 weeks.}$$

$$\text{So, } 100 + \int_0^{15} n'(t) dt = 100 + (n(15) - 100) = n(15), \text{ represents the total bee population after 15 weeks.}$$

Problem 60. For a particle moving along a line, the velocity function (in meters per second) is $v(t) = t^2 - 2t - 8$.

Find (a.) the displacement and (b.) the distance traveled by the particle during the time interval $1 \leq t \leq 6$.

a. Displacement:

$$\int_1^6 (t^2 - 2t - 8) dt$$

$$= \left[\frac{1}{3}t^3 - t^2 - 8t \right]_1^6$$

$$= (72 - 36 - 48) - \left(\frac{1}{3} - 1 - 8 \right) = -\frac{10}{3}m.$$

b. Distance traveled:

Observe that: $t^2 - 2t - 8 = (t - 4)(t + 2)$.

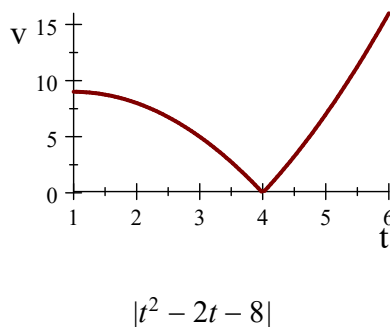
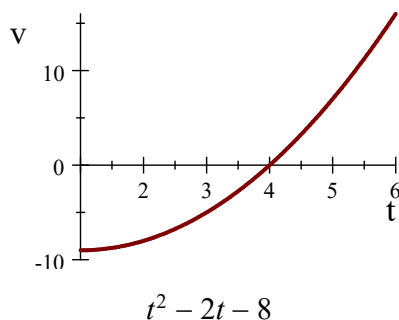
So it is negative on $1 \leq t < 4$, and nonnegative on $4 \leq t < 6$.

Therefore, we want to calculate $\int_1^6 |t^2 - 2t - 8| dt$

$$= \int_1^4 (-t^2 + 2t + 8) dt + \int_4^6 (t^2 - 2t - 8) dt$$

$$= \left[-\frac{1}{3}t^3 + t^2 + 8t \right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t \right]_4^6$$

$$= \left(-\frac{64}{3} + 16 + 32 \right) - \left(-\frac{1}{3} + 1 + 8 \right) + (72 - 36 - 48) - \left(\frac{64}{3} - 16 - 32 \right) = \frac{98}{3}m.$$



Problem 62 The acceleration function $a(t) = 2t + 3$ (in m/s^2) and the initial velocity $v(0) = -4$ are defined for a particle moving along a line. Find (i) the velocity at time t and (ii) the distance traveled during the time interval $0 \leq t \leq 3$.

i. $v'(t) = a(t) = 2t + 3$

$$v(t) = \int a(t) = t^2 + 3t + C$$

Noting the initial condition, we can solve for C : $v(0) = C = -4$.

Therefore, $v(t) = t^2 + 3t - 4$.

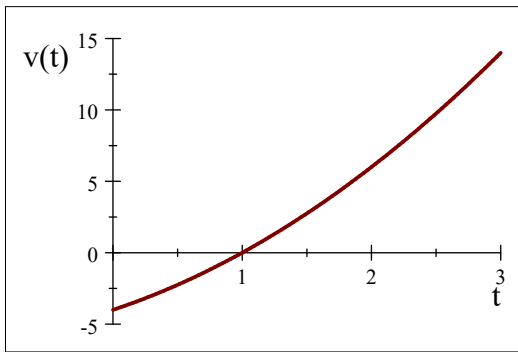
ii. Distance traveled: Since we have $v(t) = t^2 + 3t - 4$ over $0 \leq t \leq 3$, ...

$$\int_0^3 |t^2 + 3t - 4| dt = \int_0^1 (t+4)(t-1) |dt$$

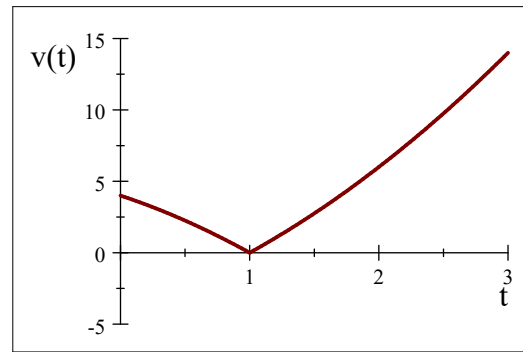
$$= \int_0^1 (-t^2 - 3t + 4) dt + \int_1^3 (t^2 + 3t - 4) dt$$

$$= \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t \right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t \right]_1^3$$

$$= \left(-\frac{1}{3} - \frac{3}{2} + 4 \right) + \left(9 + \frac{27}{2} - 12 \right) - \left(\frac{1}{3} + \frac{3}{2} - 4 \right) = \frac{89}{6} \approx 14.8 \text{ meters.}$$



Graph of $t^2 + 3t - 4$



Graph of $|t^2 + 3t - 4|$