

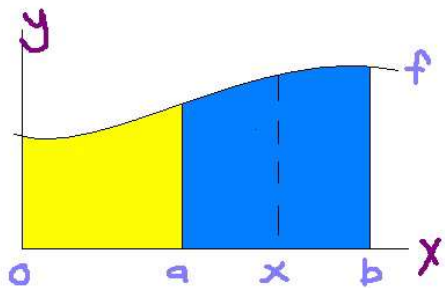
Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

5.3 - Fundamental Theorem of Calculus

Recall that $\int_a^b f(t)dt$ is a constant representing the area under the curve of f over the interval $[a, b]$. Now, let $g(x) = \int_a^x f(t)dt$.

Notice the variable x in the upper bound of integration, and that we've just created a **function** $g(x)$ that represents the area under the curve of f over the interval $[a, x]$, for any given x .



Now suppose f is continuous on $[a, b]$, with $a \leq x \leq b$:

Fundamental Theorem of Calculus (FTC)

♦ **FTC1:** $g(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) = f(x)$.

♦ **FTC2:** $\int_a^b f(x)dx = F(b) - F(a)$, where F is ANY antiderivative of f (i.e., $F' = f$).

Alternate Formulation of FTC:

♦ **FTC1:** $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.

♦ **FTC2:** $\int_a^b F'(x)dx = F(b) - F(a)$.

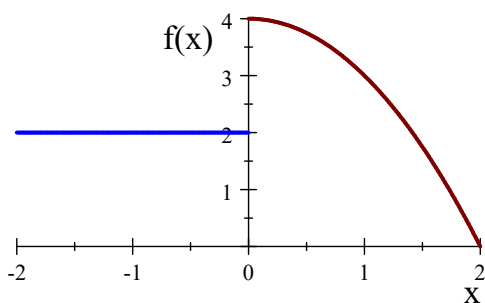
Notation you'll need: $\int_a^b F'(x)dx = [F(x)]_a^b := F(x)|_a^b := F(b) - F(a)$.

More Generally:

$\frac{d}{dx} \int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$, because of the chain rule.

And even more generally: $\frac{d}{dx} \int_{h(x)}^{g(x)} f(t)dt = \frac{d}{dx} \left(\int_{h(x)}^0 f(t)dt + \int_0^{g(x)} f(t)dt \right)$
 $= \frac{d}{dx} \left(-\int_0^{h(x)} f(t)dt + \int_0^{g(x)} f(t)dt \right) = -\frac{d}{dx} \int_0^{h(x)} f(t)dt + \frac{d}{dx} \int_0^{g(x)} f(t)dt$
 $= -f(h(x)) \cdot h'(x) + f(g(x)) \cdot g'(x)$.

Problem 44. Evaluate the integral: $\int_{-2}^2 f(x)dx$, where $f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0, \\ 4 - x^2 & \text{if } 0 < x \leq 2. \end{cases}$



$$\int_{-2}^2 f(x) dx = \int_{-2}^0 f(x) dx + \int_0^2 f(x) dx$$

$$= \int_{-2}^0 2 dx + \int_0^2 (4 - x^2) dx$$

$$= [2x]_{-2}^0 + \left[4x - \frac{1}{3}x^3 \right]_0^2$$

Recall from the review that: $[2x]_{-2}^0 = 2 \cdot 0 - 2 \cdot (-2) = 4$.

$$\text{Also: } \left[4x - \frac{1}{3}x^3 \right]_0^2 = \left(4(2) - \frac{1}{3}(2)^3 \right) - \left(4(0) - \frac{1}{3}(0)^3 \right) = \frac{16}{3}.$$

$$\text{So: } \int_{-2}^2 f(x) dx = 4 + \frac{16}{3} = \frac{28}{3}.$$

Note that f is integrable by Theorem 3 in section 5.2 because the function only has a finite number of jump discontinuities (just one) in the range of integration.

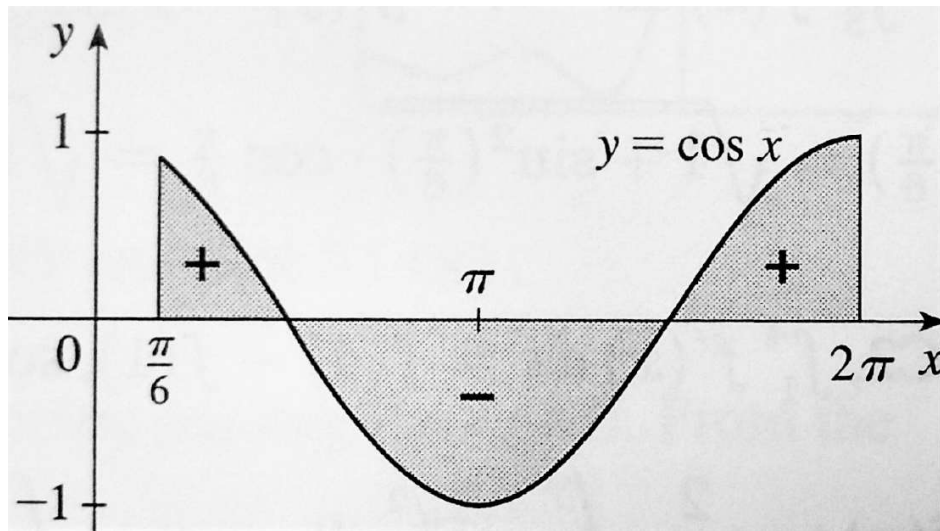
Problem 54. Evaluate $\int_{\frac{\pi}{6}}^{2\pi} \cos x dx$ and interpret it as a difference of areas. Illustrate with a sketch.

$$\int_{\frac{\pi}{6}}^{2\pi} \cos x dx = [\sin x]_{\frac{\pi}{6}}^{2\pi}$$

$$= \sin(2\pi) - \sin\left(\frac{\pi}{6}\right)$$

$$= 0 - \frac{1}{2} = -\frac{1}{2}.$$

Observe that $\cos x$ is negative between $\frac{\pi}{2}$ and $\frac{3}{2}\pi$. But elsewhere on the interval, it is positive. Therefore, our integral represents the net area under the curve for these positive and negative values of $\cos x$.

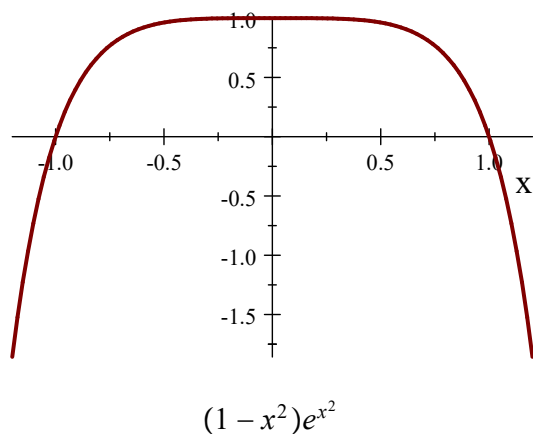


Problem 60. If $f(x) = \int_0^x (1 - t^2)e^{t^2} dt$, on what interval is the function increasing?

$f(x) = \int_0^x (1 - t^2)e^{t^2} dt$ is increasing when $f'(x) = (1 - x^2)e^{x^2}$ is positive.

Since $e^{x^2} > 0$, $f'(x) > 0$ requires that $1 - x^2 > 0$,

resulting in: $|x| < 1$, so f is increasing on $(-1, 1)$.



Problem 62. If $f(x) = \int_0^{\sin x} \sqrt{1 + t^2} dt$ and $g(y) = \int_3^y f(x) dx$, find $g''\left(\frac{\pi}{6}\right)$.

$g''\left(\frac{\pi}{6}\right)$ means the 2nd derivative evaluated at $\frac{\pi}{6}$. So let's first take our derivatives...

$$g'(y) = f(y) = \int_0^{\sin y} \sqrt{1 + t^2} dt.$$

$$g''(y) = \frac{dg'(y)}{dy} = f'(y) = \frac{d}{dy} \left[\int_0^{\sin y} \sqrt{1 + t^2} dt \right]$$

Let: $u = \sin y$ and $\frac{du}{dy} = \cos y$.

Using the chain rule: $\frac{d}{dy} = \frac{d}{du} \frac{du}{dy}$.

$$\text{So, } g''(y) = \frac{d}{du} \left[\int_0^u \sqrt{1+t^2} dt \right] \frac{du}{dy}$$

$$= \sqrt{1+u^2} \cdot \frac{du}{dy} = \sqrt{1+\sin^2 y} \cdot \cos y.$$

$$\text{So, } g''\left(\frac{\pi}{6}\right) = \sqrt{1+\sin^2\left(\frac{\pi}{6}\right)} \cdot \cos \frac{\pi}{6} = \sqrt{1+\left(\frac{1}{2}\right)^2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{5}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}.$$

Problem 69. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x) dx = 17$,
then what is the value of $f(4)$?

$$\text{By FTC2, } \int_1^4 f'(x) dx = f(4) - f(1),$$

$$\text{so } 17 = f(4) - 12 \quad \Rightarrow \quad f(4) = 17 + 12 = 29.$$

Problem 60. Find the derivative of the function: $y = \int_{1-2x}^{1+2x} t \sin t dt$

$$y = \int_{1-2x}^0 t \sin t dt + \int_0^{1+2x} t \sin t dt$$

$$\frac{d}{dx} y = -\frac{d}{dx} \int_0^{1-2x} t \sin t dt + \frac{d}{dx} \int_0^{1+2x} t \sin t dt$$

$$= -\frac{d(1-2x)}{dx} \frac{d}{d(1-2x)} \int_0^{1-2x} t \sin t dt + \frac{d(1+2x)}{dx} \frac{d}{d(1+2x)} \int_0^{1+2x} t \sin t dt \quad [\text{chain rule! } \frac{d}{dx} = \frac{d(1-2x)}{dx} \frac{d}{d(1-2x)}]$$

$$= 2 \frac{d}{d(1-2x)} \int_0^{1-2x} t \sin t dt + 2 \frac{d}{d(1+2x)} \int_0^{1+2x} t \sin t dt$$

$$= 2(1-2x) \sin(1-2x) + 2(1+2x) \sin(1+2x).$$