

Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

5.2 - Definite Integral

Sigma Notation (Σ) and Useful Sums:

Stop here (upper bound)

$$\sum_{i=1}^6 i = 1 + 2 + 3 + 4 + 5 + 6$$

Start here (lower bound)

i is called the index

$$\diamond \sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2},$$

$$\diamond \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\diamond \sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}.$$



Visual proof that $2 \sum_{i=1}^n i = n(n+1)$ or $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

Some Obvious Properties

$\sum_{i=1}^n c = nc$	$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$
$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$	

Definite Integral:

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} [f(x_1) \Delta x + f(x_2) \Delta x + \dots + f(x_n) \Delta x], \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i \Delta x.$$

If the limit exists, the function f is (Riemann) integrable. [Theorem 4]

Right Hand Integration	Left Hand Integration
$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$	$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_i) \Delta x$

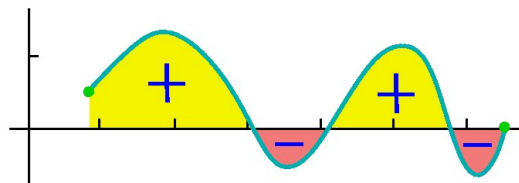
Midpoint Rule (usually the best approximation):

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(\bar{x}_i) \Delta x = [f(\bar{x}_1) + \dots + f(\bar{x}_n)] \Delta x$$

where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ = midpoint of the interval $[x_{i-1}, x_i]$.

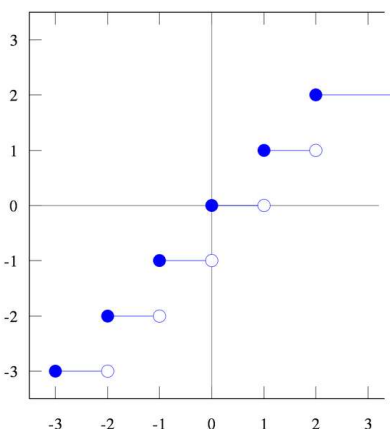
So if integration measures the area between the curve and the x-axis, what happens when the curve dips below the axis? We get negative area! This introduces the idea of net area.

Net Area: If f takes on both positive and negative values, the integral represents the **net area**, that is, the area above the curve minus the area below the curve.



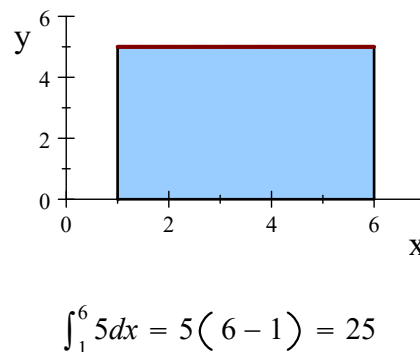
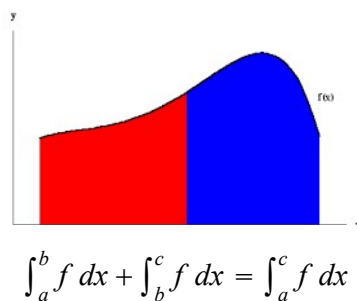
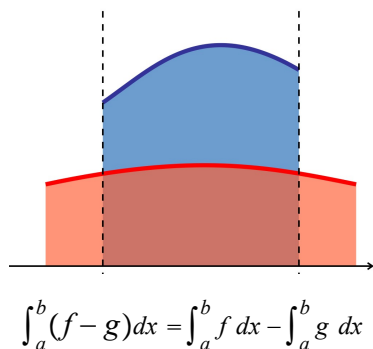
What types of functions can we integrate?

Existence of Definite Integral: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$. Recall that $\int_a^b f(x) dx$ is defined as a limit of a sum of rectangles. So, this theorem says that if the conditions above are met, that limit exists (in this context, "exists" means that $\int_a^b f(x) dx$ is equal to a non-infinite real number).



Properties of Integrals: Let c be any constant, then:

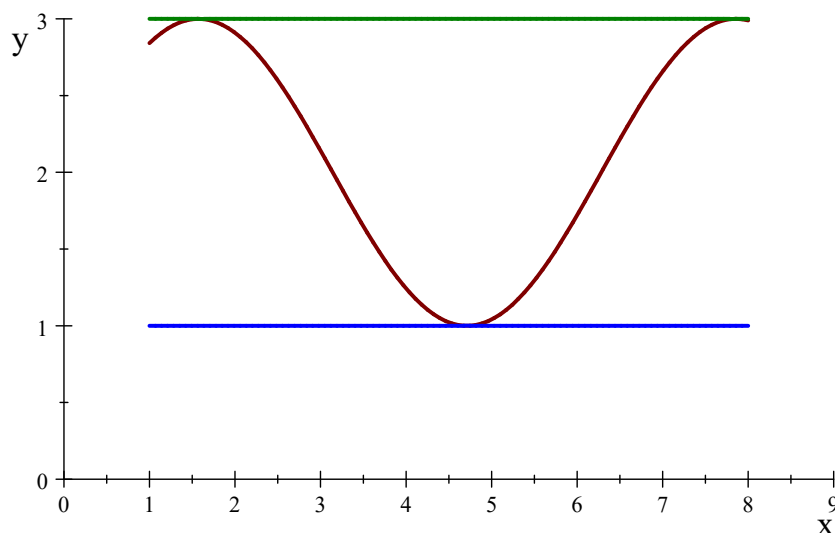
- ♦ $\int_a^b f dx = -\int_b^a f dx$,
- ♦ $\int_a^b (f - g) dx = \int_a^b f dx - \int_a^b g dx$,
- ♦ $\int_a^b c dx = c(b - a)$,
- ♦ $\int_a^a f dx = 0$,
- ♦ $\int_a^b f dx + \int_b^c f dx = \int_a^c f dx$,
- ♦ $\int_a^b (c \cdot f(x)) dx = c \int_a^b f(x) dx$.



♦ if $f \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq 0$.

♦ if $f \geq g$ for $a \leq x \leq b$, then $\int_a^b f dx \geq \int_a^b g dx$.

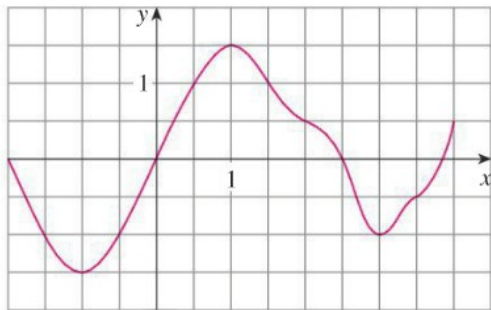
♦ if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ where $m, M \in \mathbb{R}$.



$$\text{If } 1 \leq \sin x + 2 \leq 3, \text{ then } 1(8-1) \leq \int_1^8 (\sin x + 2) dx \leq 3(8-1)$$

Problem 6. The graph of g is shown. Estimate $\int_{-2}^4 g(x) dx$ with six sub-intervals using:

(a) right endpoints, (b) left endpoints, and (c) midpoints.



Right Endpoints: $\int_{-2}^4 g(x) dx \approx [g(-1) + g(0) + \dots + g(4)] \Delta x$

$$= (-1.5 + 0 + 1.5 + 0.5 - 1 + 0.5)(1) = 0.$$

Left Endpoints: $\int_{-2}^4 g(x) dx \approx [g(-2) + g(-1) + \dots + g(3)] \Delta x$

$$= (0 - 1.5 + 0 + 1.5 + 0.5 - 1)(1) = -\frac{1}{2}.$$

Midpoints: Home Exercise!

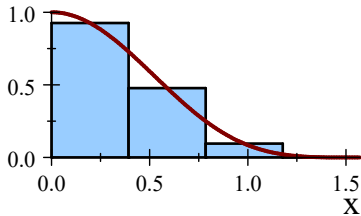
Problem 10. Use the midpoint rule with $n = 4$ to approximate the integral $\int_0^{\frac{\pi}{2}} \cos^4 x dx$.

Round the answer to four decimal places.

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}.$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx$$

$$\approx \left[\cos^4\left(\frac{\pi}{16}\right) + \cos^4\left(\frac{3\pi}{16}\right) + \cos^4\left(\frac{5\pi}{16}\right) + \cos^4\left(\frac{7\pi}{16}\right) \right] \frac{\pi}{8} \approx 0.5890.$$



$\cos^4 x$

Problem 18. Express the limit as a definite integral on the given interval.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\cos x_i}{x_i} \Delta x, \quad [\pi, 2\pi]$$

$$\int_{\pi}^{2\pi} \frac{\cos x}{x} dx.$$

Problem 24. Use the form of the definition of the integral given in Theorem 4

to evaluate the integral: $\int_0^2 (2x - x^3) dx.$

Recall Theorem 4: if f is integrable, then $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \Delta x$.

Note that $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0 + i \Delta x = \frac{2i}{n}$.

$$\text{So, } \int_0^2 (2x - x^3) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[2\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^3 \right] \frac{2}{n}$$

What do we do with this? Recall the "useful sums," and properties of sigmas.

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{8i}{n^2} - \frac{16i^3}{n^4} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n \frac{8i}{n^2} - \sum_{i=1}^n \frac{16i^3}{n^4} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{8}{n^2} \sum_{i=1}^n i - \frac{16}{n^4} \sum_{i=1}^n i^3 \right]$$

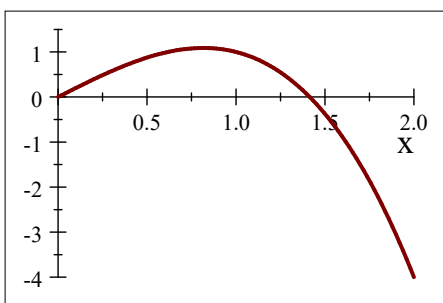
Recall you can pull a constant out of a sum, and to the sum, the n is just a constant. Yes, the limit is changing n , but for each sum, it is just a constant.

$$= \lim_{n \rightarrow \infty} \left[\frac{8}{n^2} \frac{n(n+1)}{2} - \frac{16}{n^4} \frac{n^2(n+1)^2}{4} \right]$$

$$= \lim_{n \rightarrow \infty} \left[4 \frac{n^2+n}{n^2} - 4 \frac{(n+1)^2}{n^2} \right]$$

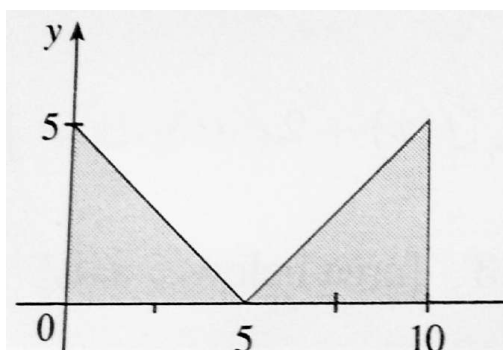
$$= \lim_{n \rightarrow \infty} \left[4 \left(1 + \frac{1}{n} \right) - 4 \left(1 + \frac{1}{n} \right)^2 \right]$$

$$= 4 \cdot 1 - 4 \cdot 1 = 0.$$



$$2x - x^3$$

Problem 40. Evaluate the integral $\int_0^{10} |x - 5| dx$ by interpreting it in terms of areas.



This function can be interpreted as the sum of the areas of the 2 shaded triangles;

$$\text{that is, } 2 \cdot \left(\text{Area of Triangle} \right) = 2 \left(\frac{1}{2} \right) (\text{width})(\text{height}) = 2 \left(\frac{1}{2} \right) (5)(5) = 25 \text{ units.}$$
