

Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

3.6 - Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x \cdot \ln e} \cdot \frac{d}{dx}x = \frac{1}{x}.$$

But more generally for any function $f(x)$ and log base a :

$$\frac{d}{dx}(\log_a(f)) = \frac{d}{dx}\left(\frac{\ln f}{\ln a}\right) = \frac{1}{\ln a} \frac{d}{dx} \ln f = \frac{1}{\ln a} \cdot \left(\frac{1}{f} \cdot f'\right) = \frac{f'}{f \cdot \ln a}.$$

(using the log rule from algebra, and the chain rule)

$$\text{For example } \frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a} \cdot \frac{d}{dx}x = \frac{1}{x \cdot \ln a}.$$

More Notation/Examples:

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx} \text{ or } \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}.$$

If we have $f(x) = \ln|x|$, then since $f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$, it must be that $f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$, therefore

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \text{ when } x \neq 0.$$

This is also more generally true in that: $\frac{d}{dx} \ln|f(x)| = \frac{f'(x)}{f(x)}$, when $f(x) \neq 0$.

Steps in Logarithmic Differentiation of an equation: $y = f(x)$

- ♦ Take natural logarithm of both sides and use the Laws of Logarithms to simplify.
- ♦ Differentiate implicitly with respect to x .
- ♦ Solve the resulting equation for y' .

Note the following weird fact: $e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$, for $n \in \mathbb{N}$.

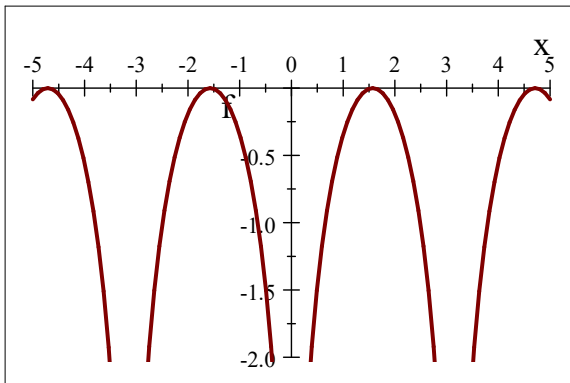
Problem 4. Differentiate the function: $f(x) = \ln[\sin^2 x]$

$$f(x) = \ln[(\sin x)^2]$$

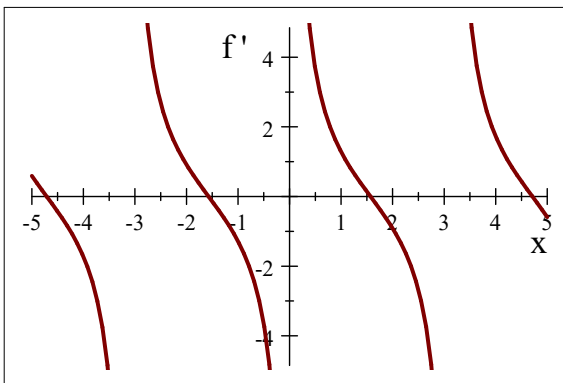
$$= 2 \ln|\sin x|.$$

$$f' = 2 \cdot \frac{1}{\sin x} \cdot \cos x \qquad \text{Recall: } \frac{d}{dx} \ln|f(x)| = \frac{f'(x)}{f(x)}, \text{ when } f(x) \neq 0.$$

$$= 2 \cot x.$$



$$f(x) = \ln[\sin^2 x]$$



$$f' = 2 \cot x.$$

Problem 21. Differentiate the function: $y = 2x \log_{10} \sqrt{x}$

$$y = 2x \log_{10} x^{\frac{1}{2}}$$

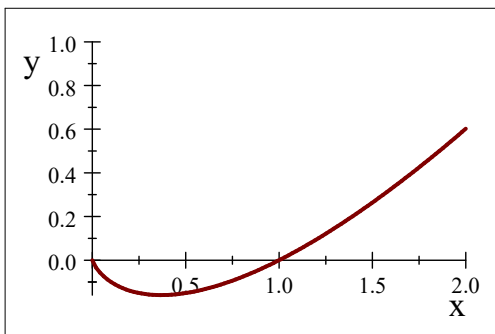
$$= 2x \cdot \frac{1}{2} \log_{10} x$$

$$= x \log_{10} x = x \left(\frac{1}{\ln 10} \ln x \right) = \frac{1}{\ln 10} x \ln x$$

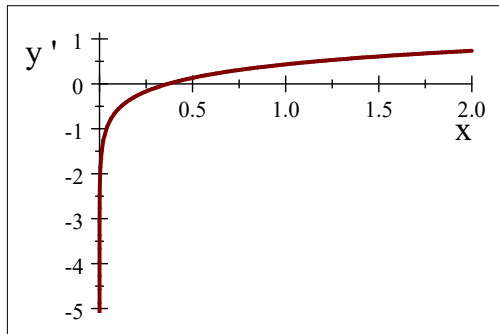
$$y' = \frac{1}{\ln 10} \left(1 \cdot \ln x + x \cdot \frac{1}{x} \right) = \frac{\ln x}{\ln 10} + \frac{1}{\ln 10}.$$

Note: $\frac{1}{\ln 10} = \frac{\ln e}{\ln 10} = \log_{10} e$, so the answer could be written as

$$\log_{10} x + \log_{10} e = \log_{10} ex.$$



$$y = 2x \log_{10} \sqrt{x}$$



$$y' = \log_{10} ex$$

Problem 30. Differentiate $f(x) = \ln(\ln(\ln x))$ and find the domain of f .

$$f' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx} \ln(\ln x)$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx} \ln x$$

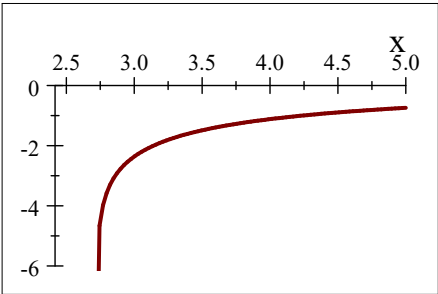
$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$Dom(f) = \{x | \ln(\ln x) > 0\}$$

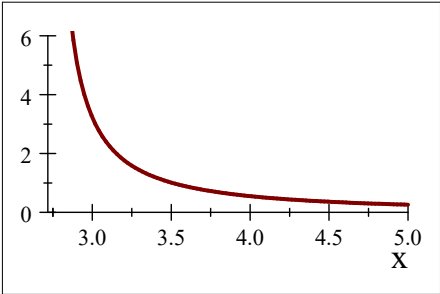
$$= \left\{x \mid \ln x > 1\right\}$$

$$= \left\{x \mid x > e\right\}$$

$$= (e, \infty).$$



$\ln(\ln(x))$



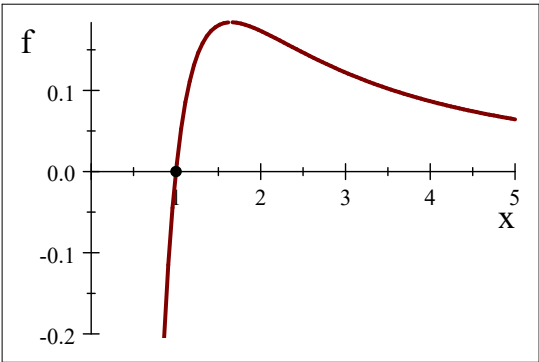
$\frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

Problem 31. If $f(x) = \frac{\ln x}{x^2}$, find $f'(1)$.

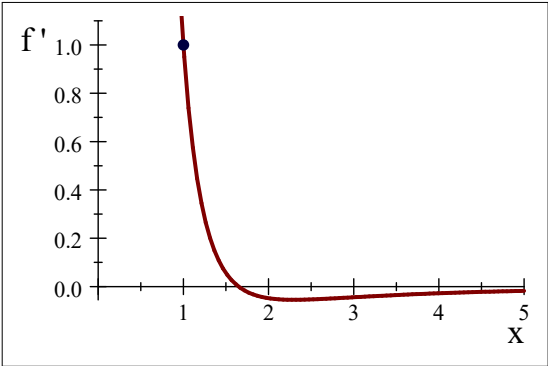
$$f' = \frac{(\frac{1}{x})x^2 - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x \ln x}{x^4}$$

$$= \frac{1 - 2 \ln x}{x^3},$$

$$\text{So } f'(1) = \frac{1 - 2 \ln 1}{1^3} = \frac{1 - 2 \cdot 0}{1} = 1.$$



$\frac{\ln x}{x^2}$



$\frac{1 - 2 \ln x}{x^3}$

Problem 48. Use logarithmic differentiation to find the derivative of: $y = (\sin x)^{\ln x}$.

$$\ln y = \ln((\sin x)^{\ln x})$$

$$\Rightarrow \ln y = \ln x \cdot \ln(\sin x) \qquad \text{(now take your derivative!)}$$

$$\Rightarrow \frac{1}{y}y' = \frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \frac{1}{\sin x} \cdot \cos x \qquad \text{(recall: } \frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx} \text{)}$$

$$\Rightarrow y' = y \left(\frac{\ln(\sin x)}{x} + \ln x \cdot \frac{\cos x}{\sin x} \right)$$

This is technically correct. However, since the problem gave us y in terms of x , (if possible) we should provide our y' in terms of x . In other words, we should eliminate the " y " in our equation.

$$\Rightarrow y' = (\sin x)^{\ln x} \left(\frac{\ln(\sin x)}{x} + \ln x \cot x \right).$$