

Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

3.4 - Chain Rule

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) := f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = (f(g))' = f'(g(x)) \cdot g'(x)$.

For example let $f := \sin(x)$ and $g := x^2$. Then $F = f(g) = \sin(x^2)$ and $F' = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot (2x)$.

Derivative in Leibniz notation: $\frac{dF}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg} \frac{dg}{dx}$.

Alternate notation: $F = f \circ g$.

Power Rule Combined with the Chain Rule: If n is any real number and $u = g(x)$ is differentiable, then $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$.
Alternative notation: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$.

Problem 17. Find the derivative of $f(x) = (x^2 + x + 1)^5(2x - 3)^4$.

We think of $g := x^2 + x + 1$ and $h := 2x - 3$, so that $f = g^5 h^4$.

By product and chain rule, we have:

$$f' = (g^5)' h^4 + g^5 (h^4)' = (5g^4 \cdot g') h^4 + g^5 (4h^3 \cdot h')$$

$$= 5(x^2 + x + 1)^4 (2x + 1) (2x - 3)^4 + (x^2 + x + 1)^5 \cdot 4(2x - 3)^3 \cdot 2$$

$$= 5(x^2 + x + 1)^4 (2x + 1) (2x - 3)^4 + 8(x^2 + x + 1)^5 (2x - 3)^3 \quad (\text{you could stop here})$$

$$= (2x - 3)^3 (x^2 + x + 1)^4 [5(2x - 3)(2x + 1) + 8(x^2 + x + 1)]$$

$$= (2x - 3)^3 (x^2 + x + 1)^4 (20x^2 - 20 - 15 + 8x^2 + 8x + 8)$$

$$= (2x - 3)^3 (x^2 + x + 1)^4 (28x^2 - 12x - 7).$$

Problem 31. Find the derivative of $y = \sin(\tan 2x)$.

Let $u = \tan 2x$. Therefore, $y = \sin u$. Note: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \cos u \cdot \frac{d}{dx}(\tan 2x) \\ &= \cos(\tan 2x) \cdot \sec^2(2x) \cdot \frac{d}{dx}(2x) \\ &= 2 \cos(\tan 2x) \sec^2(2x).\end{aligned}$$

So we actually had $y = f(u(v))$, where $f = \sin(x)$, $u = \tan(x)$, and $v = 2x$. (double chain rule!)

Problem 39. Find the derivative of $f(t) = \tan(e^t) + e^{\tan t}$.

$$\begin{aligned}y' &= \sec^2(e^t) \cdot \frac{d}{dt}(e^t) + e^{\tan t} \cdot \frac{d}{dt}(\tan t) \\ &= \sec^2(e^t) \cdot e^t + e^{\tan t} \cdot \sec^2(t) \\ &= e^t \sec^2(e^t) + e^{\tan t} \sec^2(t)\end{aligned}$$

Problem 47. Find y' and y'' of $y = \cos(x^2)$.

$$\begin{aligned}y' &= -\sin(x^2) \cdot 2x = -2x \sin(x^2) \\ y'' &= (-2) \sin(x^2) - 2x \cos(x^2) \cdot 2x \quad \text{(product and chain rule!)} \\ &= -2 \sin(x^2) - 4x^2 \cos(x^2).\end{aligned}$$

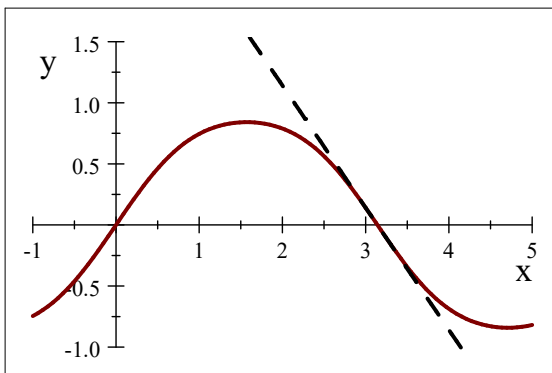
Problem 53. Find an equation of the tangent line to $y = \sin(\sin x)$ at the point $(\pi, 0)$.

$$y' = \cos(\sin x) \cdot \cos x.$$

$$\begin{aligned}\text{At } (\pi, 0), \quad y' &= \cos(\sin \pi) \cdot \cos \pi \\ &= \cos(0) \cdot (-1) = 1(-1) = -1.\end{aligned}$$

And an equation of the tangent line is

$$y - 0 = -1(x - \pi), \text{ or } y = -x + \pi.$$



$\sin(\sin x)$ and $y = -x + \pi$

Problem 73. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

$$F' = f'(3f(4f)) \cdot \frac{d}{dx}(3f(4f)) = f'(3f(4f)) \cdot 3f'(4f) \cdot \frac{d}{dx}(4f)$$

$$= f'(3f(4f)) \cdot 3f'(4f) \cdot 4f', \text{ so...}$$

$$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0)$$

$$= f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2$$

$$= 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$$

Problem 78. Find the 1,000th derivative of $f(x) = xe^{-x}$.

$$f' = e^{-x} - xe^{-x} = (1 - x)e^{-x},$$

$$f'' = -e^{-x} + (1 - x)(-e^{-x}) = (x - 2)e^{-x}.$$

$$\text{Similarly, } f''' = (3 - x)e^{-x},$$

$$f^{(4)} = (x - 4)e^{-x}, \dots, f^{(1,000)} = (x - 1000)e^{-x}.$$