

Calculus I

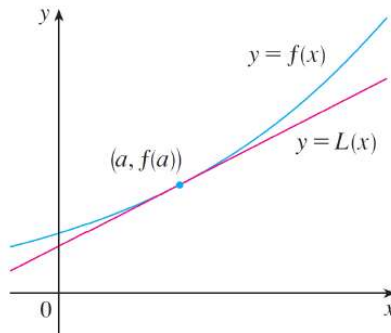
Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

3.10 - Linear Approximations and Differentials

Linearization of f at a : The linear function whose graph is the tangent line,

that is, $L(x) = f(a) + f'(a)(x - a)$.

So, $L(x) = \text{initial height} + \frac{\text{rise}}{\text{run}} \cdot \text{run} = f(x)$ approximation.



Differentials:

Consider $y = f(x)$, where f is a differentiable function.

We know how to calculate the derivative $f'(x)$. And if we look at the tangent line starting at (x_0, y_0) on the graph of $f(x)$, we can calculate the slope $f'(x_0)$. We can now define a NEW function dy , written like $dy = f'(x_0)dx$ (this should remind you of the equation for a line $y = mx$). If we were to travel along the x-axis by an amount equal to dx , then dy is the change in the height of the tangent line starting at (x_0, y_0) . Note that since we are traveling along the tangent line, and not the function itself, dy will most likely be a different value than Δy (which is the symbol representing the change in the ACTUAL function when traveling dx or Δx , which both represent the same thing). dx and dy are called **differentials**, they are approximations based on a derivative.

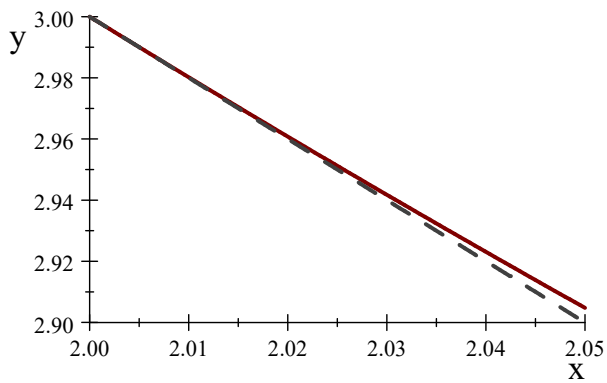
Stated another way: $dx = \Delta x$ represents the size of the interval over which you are considering the linearization of f . While $dy = \Delta L$ represents the size of the change of the linearization L of f over dx .

Problem 18. Find the differential dy and evaluate it for the given values of x and dx .

$$y = \frac{x+1}{x-1}, \quad x_0 = 2, \quad dx = 0.05$$

$$dy = f'(x_0)dx = \left(\frac{(1)(x-1) - (x+1)(1)}{(x-1)^2} \Big|_{x=2} \right) dx = -2dx$$

$$dy = -2(0.05) = -0.1.$$



$y = \frac{x+1}{x-1}$. Dashed line is the Linearization

Problem 22. Compute Δy and dy for $y = e^x$ with $x_0 = 0$ and $dx = \Delta x = 0.5$. Then sketch a diagram showing the line segments with lengths dx , dy , and Δy .

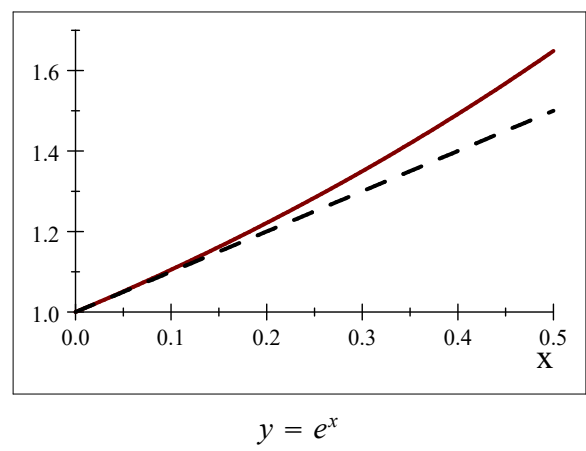
$$\Delta y = f(0.5) - f(0)$$

$$= \sqrt{e} - 1 \approx 0.65.$$

$$dy = \dots$$

$$= e^x dx$$

$$= e^0(0.5) = 0.5.$$



Problem 26. Use a linear approximation (differentials) to estimate the value for $\frac{1}{4.002}$. (Imagine you didn't have a calculator. What function does this number resemble?)

$$y = \frac{1}{x} \text{ near } x = 4.$$

$$\Rightarrow dy = -\frac{1}{x^2} dx.$$

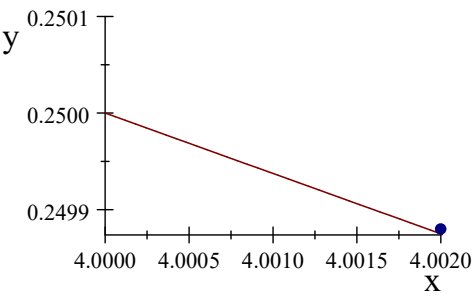
When $x = 4$ and $dx = 0.002$,

$$dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}, \text{ so}$$

$$\frac{1}{4.002} \approx f(4) + dy$$

$$= \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875. \quad \blacksquare$$

Which is off by about .001%



$\frac{1}{x}$. Black dot is the approximation

Problem 30. Explain, in terms of linear approximations or differentials, why the approximation $(1.01)^6 \approx 1.06$ is reasonable.

If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope $y' = 6(1)^5 = 6$.

And if the change in x is 0.01, then the change in y on the tangent line is $dy = 6(0.01) = 0.06$.

So approximating $(1.01)^6$ with 1.06 is reasonable.