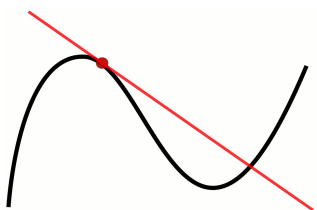


Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

2.7 - Derivatives and Rates of Change



Tangent: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists.

Or equivalently, $m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$.

Derivative of a Function: The **derivative of a function f at a number a** , denoted by $f'(a)$ is $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, **if this limit exists in $(-\infty, \infty)$!**

Velocity: If x is a position function, then at a we have the velocity: $v(a) = x'(a) = \lim_{h \rightarrow 0} \frac{x(a+h) - x(a)}{h}$.

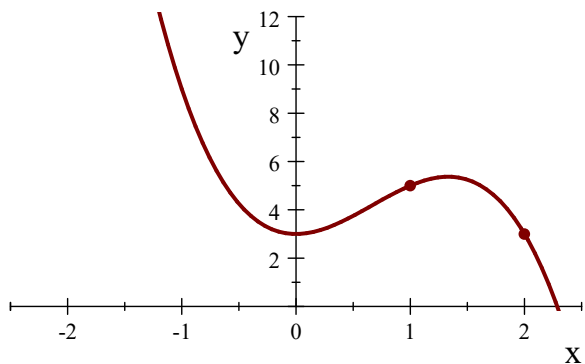
The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a .

Instantaneous Rate of Change: $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Problem 9a. Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(3 + 4(a+h)^2 - 2(a+h)^3) - (3 + 4a^2 - 2a^3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h} = \lim_{h \rightarrow 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} \\ f'(a) &= \lim_{h \rightarrow 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h} = \lim_{h \rightarrow 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2. \end{aligned}$$



Problem 9b. Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.

At $(1, 5)$:

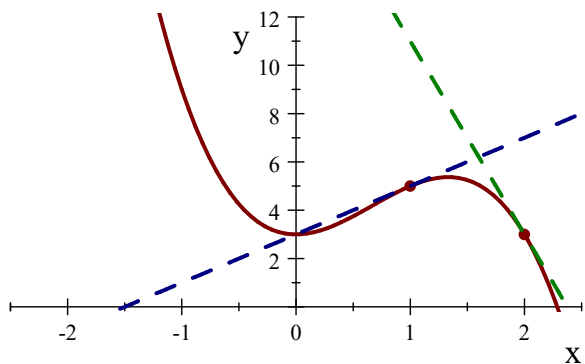
$m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is...

$y - 5 = 2(x - 1) \Leftrightarrow y = 2x + 3$.

At $(2, 3)$:

$m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is...

$y - 3 = -8(x - 2) \Leftrightarrow y = -8x + 19$.



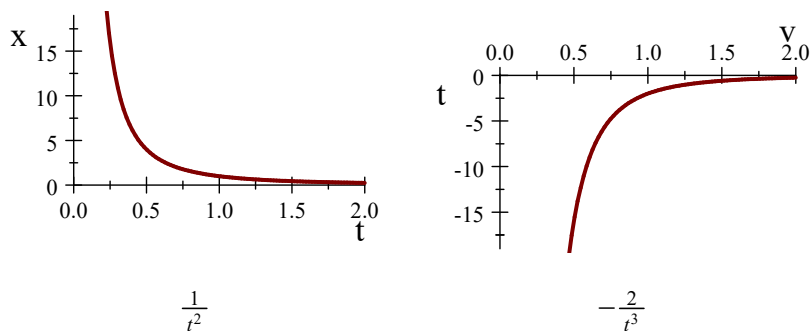
Problem 15a. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $x = \frac{1}{t^2}$, where t is measured in seconds. Find the velocity of the particle at time $t = a$.

$$v(a) = x'(a) = \lim_{h \rightarrow 0} \frac{x(a+h) - x(a)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h} \\
&= \lim_{h \rightarrow 0} \frac{a^2 - (a+h)^2}{h \cdot a^2 (a+h)^2} \quad (\text{by multiplying the numerator and denominator by } a^2(a+h)^2) \\
&= \lim_{h \rightarrow 0} \frac{a^2 - (a^2 + 2ah + h^2)}{a^2 h (a+h)^2} \\
&= \lim_{h \rightarrow 0} \frac{-2ah - h^2}{ha^2 (a+h)^2} \\
&= \lim_{h \rightarrow 0} \frac{-2a - h}{a^2 (a+h)^2} \\
&= -\frac{2a}{a^2 \cdot a^2} = -\frac{2}{a^3} \frac{\text{meters}}{\text{second}}.
\end{aligned}$$

15b. Find the velocity of the particle at time $t = 1, 2, 3$.

$$v(1) = -2 \frac{m}{s}, \quad v(2) = -\frac{1}{4} \frac{m}{s}, \text{ and } v(3) = -\frac{2}{27} \frac{m}{s}.$$



Problem 31. Find $f'(a)$ of $f(x) = \sqrt{1-2x}$.

$$\begin{aligned}
f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} = \lim_{h \rightarrow 0} \frac{(\sqrt{1-2(a+h)})^2 - (\sqrt{1-2a})^2}{h(\sqrt{1-2(a+h)} + \sqrt{1-2a})} \\
&= \lim_{h \rightarrow 0} \frac{(1-2a-2h) - (1-2a)}{h(\sqrt{1-2(a+h)} + \sqrt{1-2a})} \\
&= \lim_{h \rightarrow 0} \frac{-2h}{h(\sqrt{1-2(a+h)} + \sqrt{1-2a})} = \lim_{h \rightarrow 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}}
\end{aligned}$$

$$= \frac{-2}{\sqrt{1-2a}+\sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}.$$

Problem 37. The limit $(\lim_{h \rightarrow 0} \frac{\cos(\pi+h)+1}{h})$ represents the derivative of many functions f at some numbers a .

State such an f and a :

$$\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$$

$$f = \cos x \text{ and } a = \pi \text{ or } \dots$$

$$f = \cos(x + \pi) \text{ and } a = 0, \text{ or } \dots$$

$$f = \cos(x - 2k\pi) \text{ and } a = \pi(2k + 1).$$