

Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

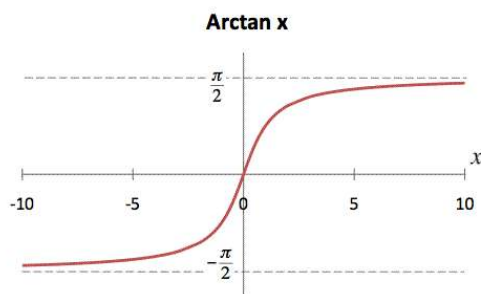
2.6 - Limits at Infinity; Horizontal Asymptotes

Definition of Asymptotic Limit: Let f be a function defined on some interval (a, ∞) . Then, $\lim_{x \rightarrow \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

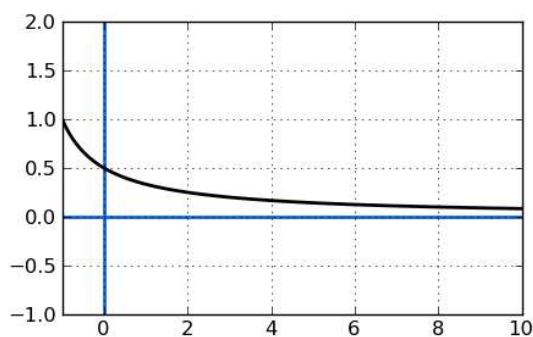
Similarly, Let f be a function defined on some interval $(-\infty, a)$. Then, $\lim_{x \rightarrow -\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently negative.

Horizontal Asymptote: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$.

Observe that: $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$ and $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$.

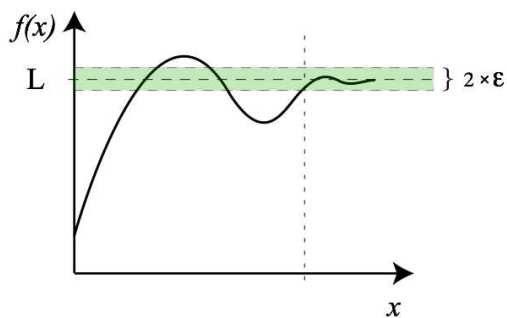


Theorem: Assuming $r > 0$ is a rational number: $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$. Also, if x^r is defined for all x (which would not be true for $r = \frac{1}{2}$ for example), we have $\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$.



$$\frac{1}{x^r}, r > 0$$

Precise Definition of Asymptotic Limit: Let f be a function defined on some interval (a, ∞) . Then $\lim_{x \rightarrow \infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number of N such that if $x > N$, then $|f(x) - L| < \varepsilon$.

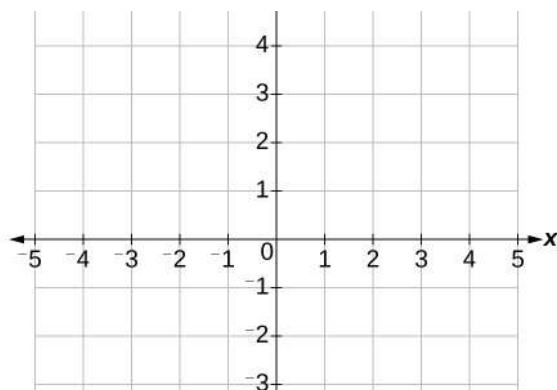


Infinity as a Limit: Let f be a function defined on some interval (a, ∞) . Then, $\lim_{x \rightarrow \infty} f(x) = \infty$ means that for every positive number M , there is a corresponding positive number N such that if $x > N$, then $f(x) > M$.

Problem 9. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$f(0) = 3, \quad \lim_{x \rightarrow 0^-} f(x) = 4, \quad \lim_{x \rightarrow 0^+} f(x) = 2,$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow 4^-} f(x) = -\infty, \quad \lim_{x \rightarrow 4^+} f(x) = \infty.$$



Problem 14. Evaluate the limit and **justify each step** by indicating the appropriate properties of limits.

$$\lim_{x \rightarrow \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$

[Limit Law 11 (radicand > 0 ? We shall see.)]

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{12 - \frac{5}{x^2} + \frac{2}{x^3}}{\frac{1}{x^3} + \frac{4}{x} + 3}}$$

[divide numerator/denominator by x^3]

$$= \sqrt{\frac{\lim_{x \rightarrow \infty} \left(12 - \frac{5}{x^2} + \frac{2}{x^3}\right)}{\lim_{x \rightarrow \infty} \left(\frac{1}{x^3} + \frac{4}{x} + 3\right)}}$$

[Limit Law 5, (limit of denominator $= 0$? We shall see.)]

$$= \sqrt{\frac{\lim_{x \rightarrow \infty} (12) - \lim_{x \rightarrow \infty} \left(\frac{5}{x^2}\right) + \lim_{x \rightarrow \infty} \left(\frac{2}{x^3}\right)}{\lim_{x \rightarrow \infty} \left(\frac{1}{x^3}\right) + \lim_{x \rightarrow \infty} \left(\frac{4}{x}\right) + \lim_{x \rightarrow \infty} (3)} \quad \left[\text{Limit Laws 1 and 2} \right]$$

$$= \sqrt{\frac{12 - 5 \lim_{x \rightarrow \infty} \left(\frac{1}{x^2}\right) + 2 \lim_{x \rightarrow \infty} \left(\frac{1}{x^3}\right)}{\lim_{x \rightarrow \infty} \left(\frac{1}{x^3}\right) + 4 \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) + 3} \quad \left[\text{Limit Laws 3 and 7} \right]$$

$$= \sqrt{\frac{12 - 5(0) + 2(0)}{0 + 4(0) + 3}} \quad \left[\text{Theorem 5 of Section 2.6} \right]$$

(Here we see that our first and 3rd step above were justified.)

$$= \sqrt{\frac{12}{3}} = \sqrt{4} = 2.$$

Problem 21. Find the limit or show that it does not exist.

$$\lim_{x \rightarrow \infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{(2x^2+1)^2}{x^4}}{\frac{(x-1)^2(x^2+x)}{x^4}}$$

$$= \lim_{x \rightarrow \infty} \frac{\left[\frac{2x^2+1}{x^2} \right]^2}{\left[\frac{x^2-2x+1}{x^2} \right] \left[\frac{x^2+x}{x^2} \right]}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(2 + \frac{1}{x^2}\right)^2}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right) \left(1 + \frac{1}{x}\right)}$$

$$= \frac{(2+0)^2}{(1-0+0)(1+0)} = 4.$$

Problem 23. Find the limit or show that it does not exist: $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^6-6}}{x^3+1}.$

$$= \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{9x^6-6}}{x^3}}{\frac{x^3+1}{x^3}}$$

$$= \frac{\lim_{x \rightarrow \infty} \sqrt{\frac{9x^6-6}{x^6}}}{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^3}\right)} \quad \left[\text{since } x^3 = \sqrt{x^6} \text{ for } x > 0 \right]$$

$$= \frac{\lim_{x \rightarrow \infty} \sqrt{9 - \frac{6}{x^6}}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \left(\frac{1}{x^3} \right)} = \frac{\sqrt{\lim_{x \rightarrow \infty} 9 - 6 \lim_{x \rightarrow \infty} \left(\frac{1}{x^6} \right)}}{1 + 0}$$

$$= \sqrt{9 - 0} = 3.$$

Problem 35. Find the limit or show that it does not exist: $\lim_{x \rightarrow \infty} \frac{1 - e^x}{1 + 2e^x}$.

$$= \lim_{x \rightarrow \infty} \frac{\frac{1 - e^x}{e^x}}{\frac{1 + 2e^x}{e^x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 2}$$

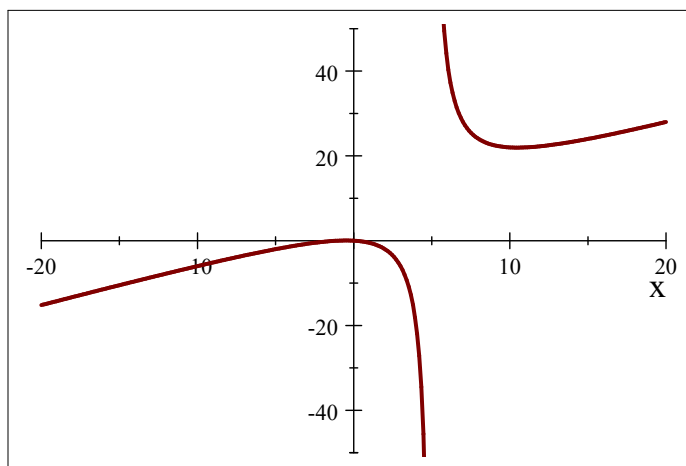
$$= \frac{0 - 1}{0 + 2} = -\frac{1}{2}.$$

Problem 45. Find the horizontal and vertical asymptotes of the curve $y = \frac{x^3 - x}{x^2 - 6x + 5}$. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

$$\frac{x^3 - x}{x^2 - 6x + 5} = \frac{x(x^2 - 1)}{(x - 5)(x - 1)}$$

$$= \frac{x(x - 1)(x + 1)}{(x - 5)(x - 1)} = \frac{x(x + 1)}{x - 5}$$

vertical at $x = 5$, and no horizontal since highest power of x is greater in numerator.



$$y = \frac{x^3 - x}{x^2 - 6x + 5}$$