

Calculus I

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

2.3 - Calculating Limits Using the Limit Laws

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then:

$$\diamond \lim_{x \rightarrow a} [f \pm g] = \lim_{x \rightarrow a} f \pm \lim_{x \rightarrow a} g \quad [\text{Limit Laws 1 and 2}]$$

$$\diamond \lim_{x \rightarrow a} [cf] = c \lim_{x \rightarrow a} f \quad [\text{Limit Law 3}]$$

$$\diamond \lim_{x \rightarrow a} [f \cdot g] = \lim_{x \rightarrow a} f \cdot \lim_{x \rightarrow a} g \quad [\text{Limit Law 4}]$$

$$\diamond \lim_{x \rightarrow a} \frac{f}{g} = \frac{\lim_{x \rightarrow a} f}{\lim_{x \rightarrow a} g}, \text{ if } \lim_{x \rightarrow a} g \neq 0 \quad [\text{Limit Law 5}]$$

$$\diamond \lim_{x \rightarrow a} [f]^n = \left[\lim_{x \rightarrow a} f \right]^n, \text{ where } n \text{ is a positive integer.} \quad [\text{Limit Law 6}]$$

$$\diamond \lim_{x \rightarrow a} c = c \quad \diamond \lim_{x \rightarrow a} x = a \quad [\text{Limit Laws 7/8}]$$

$$\diamond \lim_{x \rightarrow a} x^n = a^n, \text{ where } n \text{ is a positive integer.} \quad [\text{Limit Law 9}]$$

$$\diamond \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}, \text{ where } n \text{ is a positive integer} \quad [\text{Limit Law 10}]$$

(when n is even, we also need $a \geq 0$ for real numbers)

$$\diamond \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \text{ where } n \text{ is a positive integer.} \quad [\text{Limit Law 11}]$$

(when n is even, we also need $\lim_{x \rightarrow a} f(x) > 0$ for real numbers)

Direct Substitution Property: If f is a polynomial or a rational function (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$),

and a is in the domain of f , then $\lim_{x \rightarrow a} f(x) = f(a)$.

(this is because these types of functions are continuous on their domain)

$$\diamond \text{ If } f(x) = g(x) \text{ when } x \neq a, \text{ then } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x), \text{ provided the limits exist.}$$

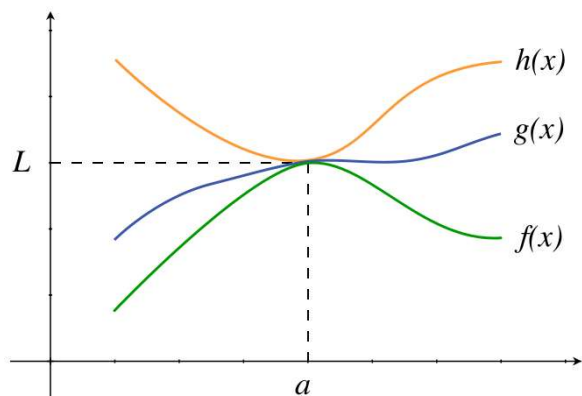
$$\diamond \lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x). \text{ (two sided limit)}$$

$$\diamond \text{ If } f(x) \leq g(x) \text{ when } x \text{ is near } a \text{ (except possibly at } a) \text{ and}$$

the limits of f and g both exist as $x \rightarrow a$, then: $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$, when x is near a (except possibly at a),

and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then: $\lim_{x \rightarrow a} g(x) = L$.



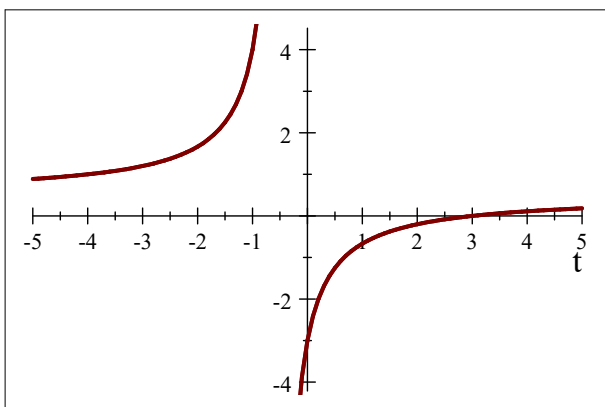
A.K.A.: The Two Policemen Theorem:

Problem 15. Evaluate the limit, if it exists: $\lim_{t \rightarrow -3} \frac{t^2-9}{2t^2+7t+3}$

$$= \lim_{t \rightarrow -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)}$$

$$= \lim_{t \rightarrow -3} \frac{t-3}{2t+1}$$

$$= \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}.$$



$$\frac{t^2-9}{2t^2+7t+3}$$

Problem 19. Evaluate the limit, if it exists: $\lim_{x \rightarrow -2} \frac{x+2}{x^3+8}$.

Does denominator have factor of $x + 2$? Polynomial division!

$$x^3 + 8 = (x + 2)x^2 + (-2x^2 + 8) \quad (*)$$

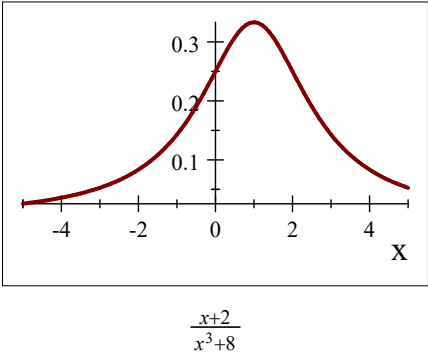
$$\begin{aligned}
 -2x^2 + 8 &= (x + 2)(-2x) + (4x + 8) \\
 &= (x + 2)(-2x) + 4(x + 2) = (x + 2)(-2x + 4).
 \end{aligned}$$

Substituting back into (*). $\qquad x^3 + 8 = (x + 2)x^2 + (x + 2)(-2x + 4) = (x + 2)(x^2 - 2x + 4).$

So, $\qquad \lim_{x \rightarrow -2} \frac{x+2}{x^3+8} = \lim_{x \rightarrow -2} \frac{x+2}{(x+2)(x^2-2x+4)}$

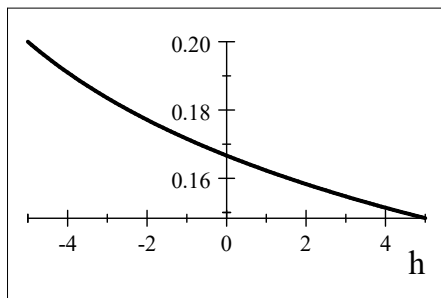
$= \lim_{x \rightarrow -2} \frac{1}{x^2-2x+4} \qquad (-2 \text{ is in the domain of this rational function})$

$= \frac{1}{4+4+4} = \frac{1}{12}.$



Problem 21. Evaluate the limit, if it exists: $\lim_{h \rightarrow 0} \frac{\sqrt{9+h}-3}{h}.$

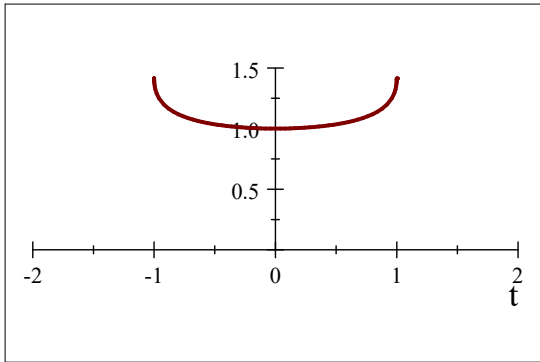
$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \right) \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} \qquad (\text{inverse "difference of squares": } (a-b)(a+b) = a^2-b^2) \\
 &= \lim_{h \rightarrow 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{9+h}+3)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{9+h}+3} \\
 &= \frac{1}{3+3} = \frac{1}{6}.
 \end{aligned}$$



$$\frac{\sqrt{9+h}-3}{h}$$

Problem 25. Evaluate the limit, if it exists: $\lim_{t \rightarrow 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

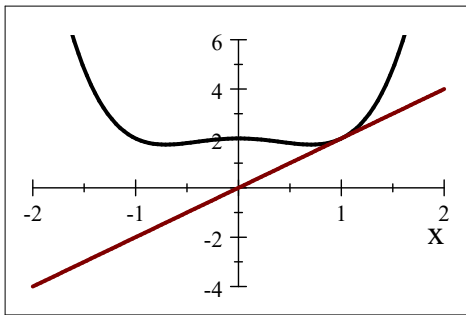
$$\begin{aligned} &= \lim_{t \rightarrow 0} \left(\frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} \right) \\ &= \lim_{t \rightarrow 0} \frac{(\sqrt{1+t})^2 - (\sqrt{1-t})^2}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t \rightarrow 0} \frac{(1+t) - (1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t \rightarrow 0} \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})} \\ &= \lim_{t \rightarrow 0} \frac{2}{(\sqrt{1+t}+\sqrt{1-t})} \\ &= \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1. \end{aligned}$$



$$\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

Problem 38. If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x , evaluate $\lim_{x \rightarrow 1} g(x)$.

$$\begin{aligned} \lim_{x \rightarrow 1} (2x) &= 2(1) = 2. \\ \text{and } \lim_{x \rightarrow 1} (x^4 - x^2 + 2) &= 1^4 - 1^2 + 2 = 2. \\ \lim_{x \rightarrow 1} g(x) &= 2 \text{ by the squeeze theorem.} \end{aligned}$$



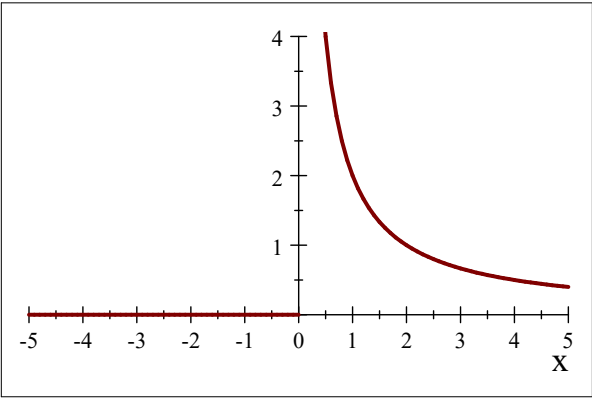
$2x$ and $x^4 - x^2 + 2$

Problem 45. Evaluate: $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} + \frac{1}{|x|} \right)$

Since $|x| = -x$ for $x < 0$,

$$\lim_{x \rightarrow 0^-} \left(\frac{1}{x} + \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{x} \right)$$

$$= \lim_{x \rightarrow 0^-} 0 = 0.$$



$$\frac{1}{x} + \frac{1}{|x|}$$