

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

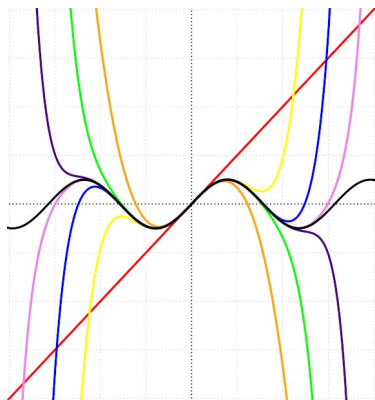
11.10 - Taylor and MacLaurin Series

Taylor Series of f at a : If f has a power series representation (can be written as $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ when $|x-a| < R$), then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

In other words,
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots$$

Maclaurin Series: is simpler and very common. It is just a Taylor series of f at 0:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$



Note: Not all functions can be represented by a power series. How do we determine if f has a power series representation?

Let $T_n(x) := \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$. This is a polynomial (partial sum) which consists of the first $n+1$ terms of the Taylor series. It is referred to as the **n th-degree Taylor polynomial of f at a** . With this notation, we hope to represent f as $f(x) = \lim_{n \rightarrow \infty} T_n(x)$.

Let's also define the **remainder** of the Taylor series as $R_n(x) := f(x) - T_n(x)$. Therefore, $f(x) = T_n(x) + R_n(x)$.

Taylor Series Representation Theorem: If $\lim_{n \rightarrow \infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$. [Theorem 8]

How do we show that $\lim_{n \rightarrow \infty} R_n(x) = 0$? We will need the following facts:

♦ **Taylor's Inequality Theorem:** If (for some n) we have $|f^{(n+1)}(x)| \leq M$

for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the

$$\text{inequality } |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for } |x-a| \leq d \quad [\text{Formula 9}].$$

♦ We'll also need that $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ for every real number x . [Equation 10]

You will see examples of how we use these facts to show $\lim_{n \rightarrow \infty} R_n(x) = 0$, and ultimately to then show that a function f is represented by its Taylor series in the problems below.

Useful Consequences of Above:

If you were to take the Maclaurin series of the following functions, you would find the following representations (do so as an exercise for one of these):

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x, \text{ so } e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}. \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x. \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x. \qquad \text{[Equation 16]} \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1. \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \text{ for } |x| < 1. \\ f(x) &:= (1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots \end{aligned}$$

The last equation is called the **binomial series**, and $\binom{k}{n}$ (pronounced "k choose n") are the binomial coefficients. It is valid for any real numbers k and $|x| < 1$.

When power series are **added or subtracted**, we can do this the same way we do for polynomials. **Multiplication and division** also work the same as polynomial multiplication and division, except in each of these cases we restrict ourselves to the first few terms, enough to get the accuracy we wish to achieve.

Problem #4 Find the Taylor series for f centered at 4 if $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$. What is the radius of convergence of the Taylor series?

Since $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$, equation 6 in the text gives the Taylor series

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!} (x-4)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)n!} (x-4)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} (x-4)^n, \text{ which is the Taylor series for } f \text{ centered at 4.} \end{aligned}$$

Apply the ratio test to find the radius of convergence R .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-4)^{n+1}}{3^{n+1} (n+2)} \cdot \frac{3^n (n+1)}{(-1)^n (x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-4)(n+1)}{3(n+2)} \right| \\ &= |x-4| \lim_{n \rightarrow \infty} \frac{n+1}{3n+6} \\ &= \frac{1}{3} |x-4| < 1 \\ \Rightarrow |x-4| &< 3 \end{aligned}$$

So $R = 3$.

Problem #8 Find the Maclaurin series for $f(x) = e^{-2x}$. [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Find the associated radius of convergence.

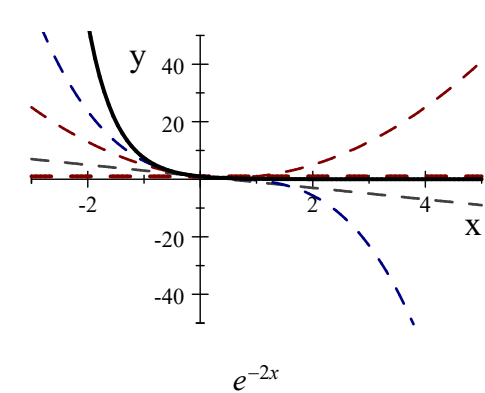
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-2x}	1
1	$-2e^{-2x}$	-2
2	$4e^{-2x}$	4
3	$-8e^{-2x}$	-8
4	$16e^{-2x}$	16
$\vdots$	$\vdots$	$\vdots$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum \frac{(-2)^n}{n!} x^n.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n x^n} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0$$

$$< 1 \text{ for all } x, \text{ so } R = \infty.$$



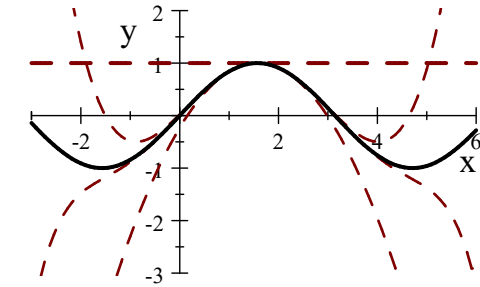
Problem #18 Find the Taylor series for $f(x) = \sin x$ centered at $a = \frac{\pi}{2}$. [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Find the associated radius of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{2})$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
 f(x) = \sin x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{2})}{k!} \left(x - \frac{\pi}{2}\right)^k \\
 &= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \frac{\left(x - \frac{\pi}{2}\right)^6}{6!} + \dots \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}.
 \end{aligned}$$

Find the associated radius of convergence.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\left(x - \frac{\pi}{2}\right)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{\left(x - \frac{\pi}{2}\right)^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{\left|x - \frac{\pi}{2}\right|^2}{2n+2} \\
 &= 0 < 1 \text{ for all } x, \text{ so } R = \infty.
 \end{aligned}$$



$\sin x$ Taylor "Jet" at $a = \frac{\pi}{2}$

Problem #22 Prove that the series obtained in problem 18 (above) represents $\sin x$, for all x .

For this, we will want to use Theorem 8:

Recall: **Taylor Series Representation Theorem:** If $\lim_{n \rightarrow \infty} R_n(x) = 0$ for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$. [Theorem 8]

So, let’s fix an arbitrary real number x (since our choice is arbitrary, if we show that this choice satisfies the theorem, the result will be true for any x).

So we need to find a limitation on the size of $R_n(x)$ to use the above theorem.

Recall: **Taylor's Inequality Theorem:** If (for some n) $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$ for $|x - a| \leq d$ [Formula 9].

So, we need to calculate a bound M on the derivatives.

Since $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$.

In each case, $|f^{(n+1)}(x)| \leq 1$, so by Formula 9 with $a = \frac{\pi}{2}$ and $M = 1$, $|R_n(x)| \leq \frac{1}{(n+1)!} \left| x - \frac{\pi}{2} \right|^{n+1}$.

Thus, $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$ by Equation 10.

Recall: $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ for every real number x . [Equation 10]

So, $\lim_{n \rightarrow \infty} R_n(x) = 0$. And by Theorem 8, the series in Problem 18 "represents" $\sin x$ (is equal to the sum of its Taylor series) for all x .

Problem #34 Use a Maclaurin series from those listed in the review section to obtain the Maclaurin series for the function $f(x) = x^2 \ln(1 + x^3)$. Find the associated radius of convergence.

Recall: $g(x) := \ln(1 + x) = \sum_{n=1}^\infty (-1)^{n-1} \frac{x^n}{n}$.

So, $g(x^3) = \ln(1 + x^3) = \sum_{n=1}^\infty (-1)^{n-1} \frac{x^{3n}}{n}$,

and $f(x) = x^2 g(x^3) = x^2 \ln(1 + x^3) = \sum (-1)^{n-1} \frac{x^{3n+2}}{n}$.

Find the associated radius of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{3(n+1)+2}}{n+1} \frac{n}{x^{3n+2}} \right| = |x^3| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x^3|.$$

So we have $R = 1$.

Problem #44 Use the Maclaurin series for e^x to calculate $\frac{1}{10^9 e}$ correct to five decimal places.

$$\frac{1}{10^9 e} = e^{-\frac{1}{10}} \text{ and recall that: } e^x = \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

$$\text{So, } e^{-\frac{1}{10}} = 1 - \frac{1}{10} + \frac{\left(\frac{1}{10}\right)^2}{2!} - \frac{\left(\frac{1}{10}\right)^3}{3!} + \dots$$

Observe that if we test $T_3(x) = 1 - \frac{1}{10} + \frac{1}{2! \cdot 100} - \frac{1}{3! \cdot 1000} \approx 0.9048333$, and then look at the next term in the alternating

series: $\frac{1}{4! \cdot 10000} \approx 0.0000041667$, we find that adding this term alters the 5th decimal place.

However, $T_4(x) = 1 - \frac{1}{10} + \frac{1}{2! \cdot 100} - \frac{1}{3! \cdot 1000} + \frac{1}{4! \cdot 10000} \approx 0.9048375$, and subtracting the next term $\frac{1}{5! \cdot 100000} \approx 0.00000008333$ does not affect the fifth decimal place, so $e^{-\frac{1}{10}} \approx 0.90484$ by the alternating series estimation theorem (from 11.5).

Note what we've accomplished. We were given a constant $e^{-\frac{1}{10}}$, and we came up with a related function e^x which allowed us to estimate the constant by using our knowledge of Taylor series.

Problem #64 Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} \\ &= \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ by Equation 16.} \end{aligned}$$