

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.9 - Representations of Functions as Power Series

Power Series for $\frac{1}{1-x}$:

Observe that $(1-x)(1+x+x^2+x^3+\dots)$

$$= (1+x+x^2+x^3+\dots) - (x+x^2+x^3+\dots) = 1.$$

So, dividing both sides by $1-x$, we have: $1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Convergent for $|x| < 1$ (geometric series).

(Notice how untrue the above calculations are for $x = 2$!!)

Calculation tricks using geometric series:

$$\frac{1}{2+x} = \frac{1}{2} \frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}},$$

$$\text{and } \frac{x^3}{1-x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3}.$$

Term-by-Term Differentiation and Integration Theorem: Power series has radius of convergence $R > 0$? Then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on interval $(a-R, a+R)$ and:

$$\diamond f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a) + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$\diamond \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of the power series in the above two equations are both R .

These two equations can be rewritten as:

$$\diamond \frac{d}{dx} \left[\sum_{n=0}^{\infty} c_n (x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n (x-a)^n],$$

$$\diamond \int \left[\sum_{n=0}^{\infty} c_n (x-a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx.$$

Warning: even though this theorem indicates that the *radius of convergence* remains the same, the endpoints may change as it relates to convergence. In other words, the *interval of convergence* may change upon taking a derivative or integrating.

Problem #2 Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the series $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$? Why?

If $f(x) := \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x) dx$

$= C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by theorem 2), but may not have the same interval of convergence — it is unknown whether the series converges at the endpoints of the interval.

Problem #10 Find a power series representation for the function $f(x) = \frac{x^2}{a^3 - x^3}$ and determine the interval of convergence.

$$\begin{aligned}
 f(x) &= x^2 \frac{1}{a^3 - x^3} \\
 &= \frac{x^2}{a^3} \cdot \frac{1}{1 - \frac{x^3}{a^3}} \\
 &= \frac{x^2}{a^3} \sum_{n=0}^{\infty} \left(\frac{x^3}{a^3} \right)^n \\
 &= \sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}.
 \end{aligned}$$

The series converges when $\left| \frac{x^3}{a^3} \right| < 1 \Rightarrow |x^3| < |a^3|$

$\Rightarrow |x| < |a|$, so $R = |a|$ (we weren't told whether a was positive),

and $I = (-|a|, |a|)$.

Problem #12 Express the function $f(x) = \frac{x+2}{2x^2-x-1}$ as the sum of a power series by first using partial fractions. Find the interval of convergence.

$$f(x) = \frac{x+2}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$x + 2 = A(x - 1) + B(2x + 1).$$

Let $x = 1$ to get $3 = 3B \Rightarrow B = 1$

$$\text{and } x = -\frac{1}{2} \Rightarrow \frac{3}{2} = -\frac{3}{2}A, \text{ or } A = -1.$$

$$\text{Thus, } \frac{x+2}{2x^2-x-1} = \frac{-1}{2x+1} + \frac{1}{x-1}$$

$$= -1 \left(\frac{1}{1-(-2x)} \right) - 1 \left(\frac{1}{1-x} \right)$$

$$= -\sum_{n=0}^{\infty} (-2x)^n - \sum_{n=0}^{\infty} x^n$$

We represented f as the sum of two geometric series; the first converges for $|2x| < 1$ or $x \in (-\frac{1}{2}, \frac{1}{2})$,

and the second converges for $(-1, 1)$.

Thus, the sum converges for $x \in (-\frac{1}{2}, \frac{1}{2}) = I$.

Problem #20 Find a power series representation for the function $f(x) = \frac{x^2+x}{(1-x)^3}$ and determine the radius of convergence.

By example 5 in the text, we have: $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$, so

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (n+1)x^n\right) \Rightarrow \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} (n+1)nx^{n-1}.$$

Thus, $f(x) = \frac{x^2+x}{(1-x)^3} = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{x^2}{2} \cdot \frac{2}{(1-x)^3} + \frac{x}{2} \cdot \frac{2}{(1-x)^3}$

$$= \frac{x^2}{2} \sum_{n=1}^{\infty} (n+1)nx^{n-1} + \frac{x}{2} \sum_{n=1}^{\infty} (n+1)nx^{n-1} \quad (\text{want to bring these under a common sum})$$

$$= \sum_{n=1}^{\infty} \frac{(n+1)n}{2}x^{n+1} + \sum_{n=1}^{\infty} \frac{(n+1)n}{2}x^n$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2}x^n + \sum_{n=1}^{\infty} \frac{(n+1)n}{2}x^n \quad (\text{make the exponents on } x \text{ equal by changing the first index})$$

$$= \sum_{n=2}^{\infty} \frac{n^2-n}{2}x^n + x + \sum_{n=2}^{\infty} \frac{n^2+n}{2}x^n \quad (\text{pull out a term to make the starting } n \text{ values equal})$$

$$= x + \sum_{n=2}^{\infty} \left(\frac{n^2-n}{2} + \frac{n^2+n}{2}\right)x^n = x + \sum_{n=2}^{\infty} n^2x^n$$

$$= \sum_{n=1}^{\infty} n^2x^n \quad (\text{reabsorbing the } x)$$

Radius of convergence: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right| = \left(\frac{n+1}{n}\right)^2|x|$.

Therefore, $\lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^2|x| = |x|$.

So f has radius of convergence $R = 1$.

Problem #26 Evaluate the indefinite integral $\int \frac{t}{1+t^3} dt$ as a power series. What is the radius of convergence?

Observe that $\frac{t}{1+t^3} = t \cdot \left(\frac{1}{1-(-t^3)}\right) = t \sum_{n=0}^{\infty} (-t^3)^n$

$$= t \sum_{n=0}^{\infty} (-1)^n (t^3)^n = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}.$$

Therefore, $\int \frac{t}{1+t^3} dt = \int \sum_{n=0}^{\infty} (-1)^n t^{3n+1} dt$

$$= \sum_{n=0}^{\infty} (-1)^n \int t^{3n+1} dt$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}. \quad \text{Convergence?}$$

The (geometric) series for $\frac{1}{1+t^3}$ converges when $|-t^3| < 1 \Rightarrow |t| < 1$, so $R = 1$ for that series and also for the series $\frac{t}{1+t^3}$.

By theorem 2, the series for $\int \frac{t}{1+t^3} dt$ also has $R = 1$.