

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.8 - Power Series

A **power series** is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$ where x is a variable and the c_n 's are constants called the coefficients of the series.

A "**power series in $(x - a)$** ", also called a **power series centered at a** , is a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots$.

For problems in the section, we need to recall:

$$(n + 1)! = 1 \cdot 2 \cdot \dots \cdot (n - 1) \cdot n \cdot (n + 1) = n! \cdot (n + 1).$$

Radius of Convergence Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ there are only three possibilities:

- ♦ The series converges only when $x = a$ (in which case $\sum_{n=0}^{\infty} c_n (x - a)^n = c_0$),
- ♦ The series converges for all x , or ...
- ♦ There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

In the last case above, R is called the **radius of convergence** of the power series. Note that this interval does not include the endpoints $(a + R$ or $a - R)$. The power series may or may not converge at these points. They must be checked individually.

To determine the radius of convergence for a power series, the **Ratio Test** (or the **Root Test**) is often useful. To do this, we determine the values of x for which $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1} (x - a)^{n+1}}{c_n (x - a)^n} \right| < 1$. It is this interval which is the radius of convergence. Examples for how to do this are in the problems below.

Interval of Convergence: If we have a radius of convergence R for a power series in $(x - a)$, then the interval of convergence is $(a - R, a + R)$.

Problem #4 Find the **radius of convergence** and **interval of convergence** for the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$.

$$\text{If } a_n := \frac{(-1)^n x^n}{\sqrt[3]{n}},$$

$$\text{then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{(-1)^n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)x \sqrt[3]{n}}{\sqrt[3]{n+1}} \right| = |x| \lim_{n \rightarrow \infty} \left| \sqrt[3]{\frac{n}{n+1}} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \sqrt[3]{\frac{1}{1 + \frac{1}{n}}} = |x|.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x| < 1$, so $R = 1$.

When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the alternating series test ($b_n = \frac{1}{\sqrt[3]{n}}$ is trivially positive, decreasing, and limit is zero).

When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges since it is a p -series ($p = \frac{1}{3} \leq 1$).

Thus, the interval of convergence is $(-1, 1]$.

Problem #28 Find the **radius of convergence** and **interval of convergence** for the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$.

If $a_n := \frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$,

$$\begin{aligned} \text{then } \lim_{n\rightarrow\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\rightarrow\infty} \left| \frac{(n+1)!x^{n+1}}{1\cdot3\cdot5\cdots(2(n+1)-1)} \cdot \frac{1\cdot3\cdot5\cdots(2n-1)}{n!x^n} \right| \\ &= \lim_{n\rightarrow\infty} \left| \frac{(n+1)x}{2n+1} \right| = |x| \lim_{n\rightarrow\infty} \left| \frac{n+1}{2n+1} \right| \\ &= |x| \lim_{n\rightarrow\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{|x|}{2}. \end{aligned}$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$ converges when $|x| < 2$, so $R = 2$.

When $x = \pm 2$, $|a_n| = \frac{n!2^n}{1\cdot3\cdot5\cdots(2n-1)} = \frac{1\cdot2\cdot3\cdots n}{1\cdot3\cdot5\cdots(2n-1)} 2^n$

It would be helpful if we could compare the numerator and the denominator...

$$\begin{aligned} &= \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n-1)} \\ &> 1. \end{aligned}$$

So, $\lim_{n\rightarrow\infty} |a_n| \geq \lim_{n\rightarrow\infty} 1 > 0$, so the series diverges by the divergence test.

Thus, the interval of convergence is $(-2, 2)$.

Problem #31 If k is a positive integer, find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$.

If $a_n := \frac{(n!)^k}{(kn)!} x^n,$

$$\begin{aligned} \text{then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{[(n+1)!]^k}{[k(n+1)]!} \cdot \frac{(kn)!}{(n!)^k} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{[(n+1)!]^k (kn)!}{(n!)^{k[k(n+1)]!}} |x| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^k (kn)!}{[kn+k]!} |x| = \lim_{n \rightarrow \infty} \frac{(n+1)^k}{(kn+1) \cdot (kn+2) \cdot \dots \cdot (kn+k-1) \cdot (kn+k)} |x| \\ &= \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+1} \cdot \frac{n+1}{kn+2} \cdot \dots \cdot \frac{n+1}{kn+k} \right] |x| \\ &= \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+1} \right] \cdot \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+2} \right] \cdot \dots \cdot \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+k} \right] |x| \\ &= \left(\frac{1}{k} \right)^k |x|. \end{aligned}$$

So, we have convergence when $\left(\frac{1}{k} \right)^k |x| < 1$ or $|x| < k^k$. And the radius of convergence is $R = k^k$.