

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.7 - Strategy for Testing Series

Strategies for Testing Series Convergence:

- ♦ If the series is of the form $\sum \frac{1}{n^p}$, it is a ***p*-series**, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
- ♦ If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a **geometric series**, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
- ♦ If the series has a form that is similar to a ***p*-series** or a **geometric series**, then one of the **comparison tests** should be considered. In particular, if a_n is a rational function (e.g., $\frac{n}{n^3+4}$) or an algebraic function of n (involving roots of polynomials: $\frac{n}{\sqrt{n^3+4}}$), then the series should be compared with a ***p*-series**. The **Comparison Tests** apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the **Comparison Test** to $\sum |a_n|$ and test for absolute convergence.
- ♦ If you can see at a glance that $\lim_{n \rightarrow \infty} a_n \neq 0$, then the **Test for Divergence** should be used.
- ♦ If the series is of the form $\sum (-1)^n b_n$, then the **Alternating Series Test** is an obvious possibility.
- ♦ Series that involve factorials or other products (including a constant raised to the n th power) are often conveniently tested using the **Ratio Test**. Bear in mind that unfortunately $|\frac{a_{n+1}}{a_n}| \rightarrow 1$ as $n \rightarrow \infty$ for all ***p*-series**, all rational functions, and all algebraic functions of n . Thus the **Ratio Test** should not be used for such series.
- ♦ If the series is of the form $(a_n)^n$, then the **Root Test** may be useful.
- ♦ If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the **Integral Test** is effective (assuming the hypotheses of this test are satisfied).

Problem #2 Test the series $\sum_{n=1}^\infty \frac{(2n+1)^n}{n^{2n}}$ for convergence and divergence.

Of the form $(b_n)^n$.

$$\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(2n+1)^n}{n^{2n}} \right|} = \lim_{n \rightarrow \infty} \frac{2n+1}{n^2}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{n} + \frac{1}{n^2} \right) = 0$$

< 1 .

So the series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ converges by the root test.

Problem #18 Test the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ for convergence and divergence.

$$b_n := \frac{1}{\sqrt{n}-1}, \text{ for } n \geq 2.$$

$\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n \rightarrow \infty} b_n = 0$.

So $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the alternating series test.

Problem #38 Test the series $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$ for convergence and divergence.

Use the limit comparison test with $a_n = \sqrt[n]{2} - 1$ and $b_n = \frac{1}{n}$.

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x \rightarrow \infty} \frac{2^{\frac{1}{x}} - 1}{\frac{1}{x}} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2^{\frac{1}{x}} \cdot \ln 2 \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \left(2^{\frac{1}{x}} \cdot \ln 2\right)$$

$$= 1 \cdot \ln 2 > 0.$$

So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$.

Alternatively, observe that: $(\sqrt{a} - 1)(\sqrt{a} + 1) = a - 1$.

$$\text{And: } (\sqrt[n]{a} - 1)(a^{\frac{2}{n}} + a^{\frac{1}{n}} + 1) = a - 1.$$

$$\text{And more generally: } (\sqrt[n]{a} - 1)(a^{\frac{n-1}{n}} + \dots + a^{\frac{2}{n}} + a^{\frac{1}{n}} + 1) = a - 1.$$

$$\text{So: } \sqrt[n]{2} - 1 = \frac{1}{2^{\frac{n-1}{n}} + 2^{\frac{n-2}{n}} + \dots + 2^{\frac{1}{n}} + 1}$$

$$\geq \frac{1}{2+2+\dots+2+1} > \frac{1}{2n}.$$

And since $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$ by the comparison test.