

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.6 - Absolute Convergence and the Ratio and Root Tests

Absolutely Convergent: A series $\sum a_n$ where the sum of absolute values $\sum |a_n|$ is convergent.

Conditionally Convergent: A series $\sum a_n$ that is convergent but not absolutely convergent (for instance, $\sum (-1)^n \frac{1}{n}$, since the harmonic series is not convergent)

Ratio Test:

- ♦ If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum a_n$ is absolutely convergent (and therefore convergent).
- ♦ If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum a_n$ is divergent.
- ♦ If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the ratio test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Root Test:

- ♦ If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent (and therefore convergent).
- ♦ If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum a_n$ is divergent.
- ♦ If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Rearrangements: If $\sum a_n$ is an **absolutely** convergent series with the sum $\sum a_n = s$, then any rearrangement of the terms of $\{a_n\}$ produces the same sum. This seems reasonable. However, counterintuitively, if $\sum a_n$ is a **conditionally** convergent series and r is any real number whatsoever, then there is a rearrangement of the terms of $\{a_n\}$ that will produce a sum $\sum a_n = r$!

Problem #4 Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ is absolutely convergent, conditionally convergent, or divergent.

Ler $b_n := \frac{n}{n^2+4} > 0$, for $n \geq 1$, and observe that $\{b_n\}$ is decreasing for $n \geq 2$.

Also: $\lim_{n \rightarrow \infty} b_n = 0$, so $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ converges by the alternating series test.

To determine absolute convergence, choose $a_n = \frac{1}{n}$ to get $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{n}{n^2+4}} = \lim_{n \rightarrow \infty} \frac{n^2+4}{n^2} = \lim_{n \rightarrow \infty} \frac{1+\frac{4}{n^2}}{1} = 1 > 0$.

So, $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the limit comparison test with the harmonic series.

Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ is conditionally convergent.

Problem #26 Determine whether $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ is convergent or divergent.

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-2)^n}{n^n} \right|} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{n^n}} \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} = 0 < 1,\end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ is absolutely convergent by the root test.

Problem #30 Determine whether the series: $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)}$ is absolutely convergent, conditionally convergent, or divergent.

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}(n+1)!}{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+5)}}{\frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(n+1)!}{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+5)} \cdot \frac{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)}{2^n n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2(n+1)}{3n+5} \\ &= \lim_{n \rightarrow \infty} \frac{2+\frac{2}{n}}{3+\frac{5}{n}} = \frac{2}{3} < 1, \text{ so the series is absolutely convergent (and therefore convergent) by the ratio test.}\end{aligned}$$

Problem #32 A series $\sum a_n$ is defined by the equations $a_1 = 1$ and $a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$. Determine whether $\sum a_n$ converges or diverges.

By the recursive definition, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\begin{aligned}&= \lim_{n \rightarrow \infty} \left| \frac{\frac{2+\cos n}{\sqrt{n}} a_n}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2+\cos n}{\sqrt{n}} \right| \\ &< \lim_{n \rightarrow \infty} \frac{3}{\sqrt{n}} \\ &= 0 < 1, \text{ so the series is absolutely convergent (and therefore convergent) by the ratio test.}\end{aligned}$$