

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.5 - Alternating Series

Alternating Series: A series whose terms are alternatively positive and negative. For example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Alternating Series Test: Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \dots$, where $b_n > 0$ satisfies the two conditions:

(i) $b_{n+1} \leq b_n$ (or equivalently $\frac{b_{n+1}}{b_n} \leq 1$) for all n .

We can also show this if we change $n \rightarrow x$ and $b_n \rightarrow f(x)$,

and find $f'(x) < 0$ for $x > 1$ (see problem 26 below).

(ii) $\lim_{n \rightarrow \infty} b_n = 0$; then the series is convergent.

Estimating Sums

Alternating Series Estimation Theorem: If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies the conditions (i) and (ii) from the alternating series test, then $|R_n| = |s - s_n| \leq b_{n+1}$.

Problem #4 Test the series $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \dots$ for convergence or divergence.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}}.$$

Now, $b_n = \frac{1}{\sqrt{n+1}} > 0$,

$\{b_n\}$ is decreasing,

and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$, so the series converges by the alternating series test.

Problem #20 Test the series $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$ for convergence or divergence.

$$b_n = \frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} > 0, \text{ for } n \geq 1.$$

$\{b_n\} = \left\{ \frac{1}{\sqrt{n+1} + \sqrt{n}} \right\}$ is decreasing

and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$.

So the series $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$ converges by the alternating series test.

Problem #26 Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$ is convergent. How many terms of the series do we need to add in order to find the sum to the accuracy $|error| < 0.01$?

The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{e^n}$ satisfies (i) of the alternating series test because

$$\left(\frac{x}{e^x}\right)' = \frac{e^x(1-x)e^x}{(e^x)^2} = \frac{1-x}{e^x} < 0, \text{ for } x > 1 \text{ and}$$

(ii) Recall **Theorem 3** from section 11.2, which says that: if $\lim_{x \rightarrow \infty} \frac{x}{e^x}$ exists, then $\lim_{n \rightarrow \infty} \frac{n}{e^n} = \lim_{x \rightarrow \infty} \frac{x}{e^x}$.

Using L'Hospital's rule: $\lim_{x \rightarrow \infty} \frac{x}{e^x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$.

So the limit exists (and so we're allowed to use the theorem) and $\lim_{n \rightarrow \infty} \frac{n}{e^n} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$.

So now we can say that the series $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$ is convergent.

Now, $b_6 = \frac{6}{e^6} \approx 0.015 > 0.01$

and $b_7 = \frac{7}{e^7} \approx 0.006 < 0.01$.

So by the alternating series estimation theorem, $n = 6$ (That is, since the seventh term is less than the desired error, we need to add the first six terms to get the sum to the desired accuracy.).

Problem #34 For what values of p is the series $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$ convergent?

Let $f(x) := \frac{(\ln x)^p}{x}$ for $x \geq 2$.

(i) For what p is f decreasing? $f'(x) = \frac{p(\ln x)^{p-1} - (\ln x)^p}{x^2} = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$.

When $p < \ln x$, which is when $e^p < x$.

So f is eventually decreasing for every p .

(ii) Clearly, $\lim_{n \rightarrow \infty} \frac{(\ln n)^p}{n} = 0$ if $p \leq 0$.

Case: $p > 0$.

Recall Theorem 3 from section 11.2, which says that if $\lim_{x \rightarrow \infty} \frac{(\ln x)^p}{x}$ exists, then $\lim_{n \rightarrow \infty} \frac{(\ln n)^p}{n} = \lim_{x \rightarrow \infty} \frac{(\ln x)^p}{x}$.

So, using L'Hospital's rule: $\lim_{x \rightarrow \infty} \frac{(\ln x)^p}{x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{p(\ln x)^{p-1} \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{p(\ln x)^{p-1}}{x}$

$$\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{p(p-1)(\ln x)^{p-2}}{x} \stackrel{L'H}{=} \dots \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{p(p-1)!}{x} = 0.$$

So the limit exists (and so we're allowed to use the theorem) and $\lim_{n \rightarrow \infty} \frac{(\ln n)^p}{n} = \lim_{x \rightarrow \infty} \frac{(\ln x)^p}{x} = 0$.

So the series converges for all p (by the alternating series test).