

# Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

## 11.4 - Comparison Tests

**Comparison Test:** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms.

♦ If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum a_n$  is also convergent.

♦ If  $\sum b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum a_n$  is also divergent.

(recall the similar comparison test for integral convergence in sect. 7.8)

**Compare your given series with the following series:**

♦  $p$ -series ( $\sum \frac{1}{n^p}$  converges if  $p > 1$  and diverges otherwise).

♦ geometric series ( $\sum ar^{n-1}$  converges if  $|r| < 1$ , and diverges otherwise).

Even though the requirement in the comparison test is that  $a_n \leq b_n$  or  $a_n \geq b_n$  for all  $n$ , we can relax this requirement a bit. Observe that the convergence of a series is not affected by a finite number of terms. Therefore, at the beginning of the series, it is allowed that there be a finite number of terms not satisfying the inequality. Specifically, we need only verify that the inequality holds for  $n \geq N$ , where  $N$  is some fixed integer (i.e., eventually the inequality holds).

**Limit Comparison Test:** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ , where  $c$  is a finite number and  $c > 0$ , then either **both series converge** or **both diverge**. A simple example where both series diverge:

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ . A simple example where both series converge:  $\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$ .

**Estimating Sums:** Assume you've just shown that  $\sum a_n$  is less than the convergent series  $\sum b_n$ . Now you have a convenient way to estimate  $\sum a_n$ . Let  $A_m := s - s_m = a_{m+1} + a_{m+2} + \dots$  be the remainder for  $\sum a_n$  when taking the partial sum at  $m$ . And let  $B_m := t - t_m$  be the remainder for  $\sum b_n$ . Since  $a_n \leq b_n$  for all  $n$ , we must then have  $A_m \leq B_m$ . Using the methods for estimating  $B_m$  we learned in the previous section, we can therefore estimate  $A_m$ , and therefore  $s$ .

**Problem #2** Suppose  $\sum a_n$  and  $\sum b_n$  have positive terms and  $\sum b_n$  is known to be divergent.

a) If  $a_n > b_n$  for all  $n$ , what can you say about  $\sum a_n$ ? Why?

If  $a_n > b_n$  for all  $n$ , then  $\sum a_n$  is divergent. [This is part (ii) of the comparison test]

b) If  $a_n < b_n$  for all  $n$ , what can you say about  $\sum a_n$ ? Why?

We cannot say anything about  $\sum a_n$ .

If  $a_n < b_n$  for all  $n$  and  $\sum b_n$  is divergent, then  $\sum a_n$  could be convergent or divergent.

**Problem #28** Determine whether the series  $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$  converges or diverges.

Observe that  $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n}$  for all  $n \geq 1$ ,

so  $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$  diverges by comparison with the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ .



**Problem #39**     Prove that if  $a_n \geq 0$  and  $\sum a_n$  converges, then  $\sum a_n^2$  also converges.

Since  $\sum a_n$  converges,  $\lim_{n \rightarrow \infty} a_n = 0$ , so there exists  $N$  such that  $|a_n| < 1$  for all  $n > N$ .

But since  $a_n \geq 0$ , then  $0 \leq a_n < 1$ , for all  $n > N$ .

And:  $0 \leq a_n^2 \leq a_n$ .

Then since  $\sum a_n$  converges, so does  $\sum a_n^2$  by the comparison test.

