

Calculus II

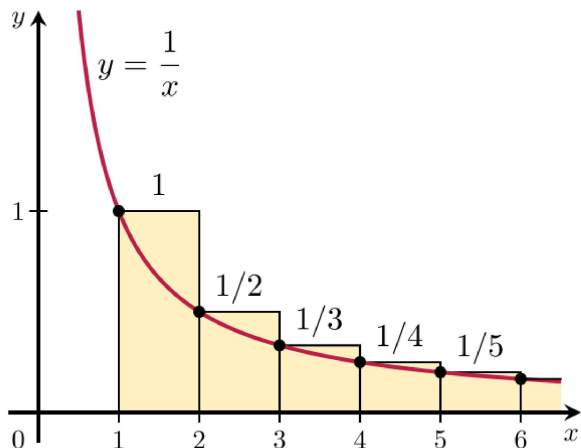
Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.3 - Integral Test and Estimates of Sums

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and assume $a_n = f(n)$. Then the series $\sum a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent. In other words:

♦ If $\int_1^\infty f(x)dx$ is convergent, then $\sum a_n$ is convergent.

♦ If $\int_1^\infty f(x)dx$ is divergent, then $\sum a_n$ is divergent.



Note that it is not necessary to start the series or the integral at $n = 1$. Also, it is not necessary that f be always decreasing, merely that it is eventually decreasing for all $x > M$, for some $M \in \mathbb{R}$.

Caution, in general $\sum_{n=1}^\infty a_n \neq \int_1^\infty f(x)dx$.

The p -series: $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(Recall that $\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$)

Estimating the Sum of a Series

Remainder: $R_n := s - s_n = a_{n+1} + a_{n+2} + \dots$

The remainder is the error made when s_n (the sum of the first n terms)

is used as an approximation to the total sum.

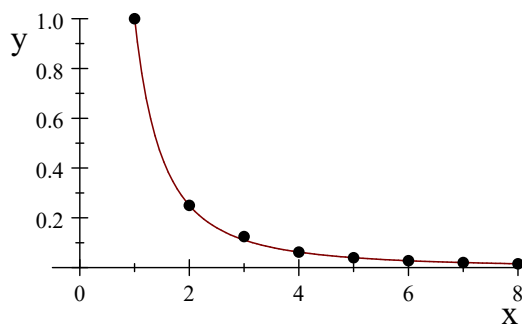
Remainder Estimate for the Integral Test:

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then $\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx$. [Equation 3]

Adding s_n to the inequality above, we get a lower and upper bound for our sum s :

$$s_n + \int_{n+1}^\infty f(x)dx \leq s \leq s_n + \int_n^\infty f(x)dx.$$

This provides a more accurate approximation to the sum of a series than the partial sum s_n does.



$$f(x) = \frac{1}{x^2}, \quad a_n = \left\{1, \frac{1}{4}, \frac{1}{9}, \dots\right\}$$

Problem #4 Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is convergent or divergent.

The function $f(x) = \frac{1}{x^5}$ is continuous, positive, and decreasing on $[1, \infty)$, so the integral test applies.

$$\int_1^{\infty} \frac{1}{x^5} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-5} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{-4}}{-4} \right]_1^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{4t^4} + \frac{1}{4} \right) = \frac{1}{4}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is also convergent by the integral test.

Problem #30 Find the values of p for which the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$ is convergent.

Can't let denominator equal zero, so $\ln n \neq 0 \Rightarrow n > 1$, and ...

What do we do about $\ln(\ln n)$?

$f(x) := \frac{1}{x \ln x [\ln(\ln x)]^p}$ is positive and continuous on $[3, \infty)$.

For $p \geq 0$, f clearly decreases on $[3, \infty)$; and for $p < 0$ it can be verified that f is ultimately decreasing.

Thus, we can apply the integral test.

$$I = \int_3^{\infty} \frac{dx}{x \ln x [\ln(\ln x)]^p} = \lim_{t \rightarrow \infty} \int_3^t \frac{[\ln(\ln x)]^{-p}}{x \ln x} dx \quad u = \ln(\ln x), \quad du = \frac{1}{x \ln x}$$

$$= \lim_{t \rightarrow \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_3^t = \lim_{t \rightarrow \infty} \left[\frac{[\ln(\ln x)]^{-p+1}}{-p+1} \right]_3^t \quad (\text{for } p \neq 1)$$

$$= \lim_{t \rightarrow \infty} \left[\frac{[\ln(\ln t)]^{-p+1}}{-p+1} - \frac{[\ln(\ln 3)]^{-p+1}}{-p+1} \right], \text{ which exists whenever } -p+1 < 0 \text{ or } p > 1.$$

If $p = 1$, then with the same u-sub we have:

$$\begin{aligned} I &= \lim_{t \rightarrow \infty} \int_3^t \frac{1}{x \ln(\ln x) \ln x} dx = \int_3^t \frac{1}{u} du = \lim_{t \rightarrow \infty} [\ln(u)]_3^t \\ &= \lim_{t \rightarrow \infty} [\ln(\ln(\ln x))]_3^t = \lim_{t \rightarrow \infty} \ln(\ln(\ln t)) - \ln(\ln(\ln 3)) = \infty. \end{aligned}$$

Therefore, $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$ converges for $p > 1$.

Problem #38 Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.

For $x > 0$, $f(x) := \frac{1}{x^5}$ is positive and continuous,

$f'(x) = -\frac{5}{x^6}$ is negative (which tells us that $f(x)$ is decreasing), and so the integral test applies.

Using Equation 3, we have $R_n \leq \int_n^{\infty} x^{-5} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{4x^4} \right]_n^t = \frac{1}{4n^4}$.

If we take $n = 5$, then $R_n \leq \frac{1}{4 \cdot 5^4} = 0.0004$.

And: $s_5 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \approx 1.036662$.

So, $s \approx s_5 \approx 1.037$.