

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

11.2 - Series

Series: If we try to add the terms of an infinite sequence $\{a_n\}$, we can write an expression of the form " $a_1 + a_2 + \dots + a_n + \dots$ " which is called an **infinite series**, or just a **series**. We notate this as $\sum_{n=1}^{\infty} a_n$ or Σa_n .

Observe that every real number can be expressed as an infinite series.

$$\begin{aligned} 0 &= 0 + 0 + \dots, & 5 &= 5 + 0 + 0 + \dots, \\ \frac{1}{3} &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots, & \pi &= 3 + \frac{1}{10} + \frac{4}{100} + \dots \end{aligned}$$

Convergence of Series: Given the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$, let s_n denote its n th **partial sum** defined as:

$s_n := \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$. Observe that s_n is a real number for each n . If the **new** sequence

$\{s_n\} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$ is convergent, and its limit $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the **original** series

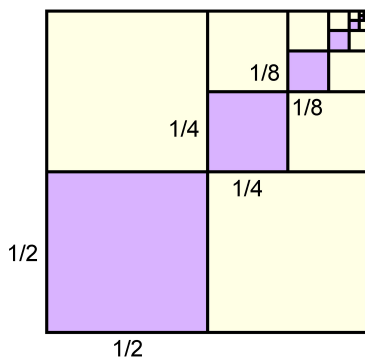
$\Sigma a_n = a_1 + a_2 + \dots$ is called **convergent**, and we write $s := a_1 + a_2 + \dots$ or $s := \sum_{n=1}^{\infty} a_n$. The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

As a mnemonic device, and to gain better understanding, observe that $\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$ is similar to how

$$\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t f(x) dx.$$

Geometric Series:

Assume $a \neq 0$, then the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots$ is called a **geometric series**. r is called its **common ratio**. It's partial sum is $s_n = \frac{a(1-r^n)}{1-r}$. If $|r| < 1$, then the geometric series is convergent, and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, then the series is divergent. (see the text for the simple derivations of these sums, which can serve as mnemonic devices).



Each of the purple squares has $1/4$ of the area of the next larger square ($1/2 \times 1/2 = 1/4$, $1/4 \times 1/4 = 1/16$, etc.). The sum of the areas of the purple squares is one third of the area of the large square.

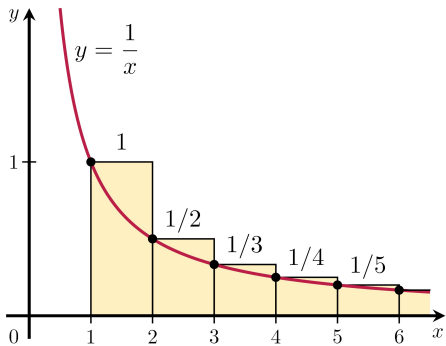
Telescopic Sum: Sometimes a series Σa_n has terms that when written out ($a_0 + a_1 + a_2 + \dots$), cancel each other leaving only a finite number of terms.

$$\begin{aligned} \text{Example: } \sum_{i=1}^n \frac{1}{i(i+1)} &= \Sigma \left(\frac{1}{i} - \frac{1}{i+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

(notice how the second term in each parentheses cancels with the first term in the next set of parentheses)



Harmonic Series: $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent! We use this series A LOT for comparison to other series. (see text for simple proof)



Convergent Series, Zero Limit Theorem: If $\sum a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$. However, it is NOT true that $\lim_{n \rightarrow \infty} a_n = 0$ implies that $\sum a_n$ is convergent. The harmonic series above is the most obvious example of this.

To avoid confusion, observe that for any series $\sum a_n$, we often refer to two different sequences: the sequence of its terms $\{a_n\}$, and the sequence of its partial sums $\{s_n\} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$. If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} s_n = s = \sum a_n$, and we can conclude that $\lim_{n \rightarrow \infty} a_n = 0$.

Test for Divergence: If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum a_n$ is divergent. (but note that this is NOT a test for convergence, only divergence)

Linearity of Convergent Series: If $\sum a_n$ and $\sum b_n$ are convergent series, then so are $\sum(a_n + b_n)$, $\sum(a_n - b_n)$, and $\sum ca_n$ (where c is a constant). We also have the following:

$$\diamond \sum ca_n = c\sum a_n \qquad \diamond \sum(a_n \pm b_n) = \sum a_n \pm \sum b_n.$$

Problem #2 Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

It means that by adding sufficiently many terms of the series the partial sums will get as close to 5 as we like.

Problem #4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ with partial sums $s_n = \frac{n^2-1}{4n^2+1}$.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n^2-1}{4n^2+1} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{n^2-1}{n^2}}{\frac{4n^2+1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1-\frac{1}{n^2}}{4+\frac{1}{n^2}} = \frac{\lim_{n \rightarrow \infty} \left(1-\frac{1}{n^2}\right)}{\lim_{n \rightarrow \infty} \left(4+\frac{1}{n^2}\right)} \quad \left(\text{since } \lim_{n \rightarrow \infty} \left(4+\frac{1}{n^2}\right) \neq 0\right) \end{aligned}$$

$$= \frac{1 - \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right)}{4 + \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right)} = \frac{1-0}{4+0} = \frac{1}{4}.$$

Problem #7 For the series $\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$, calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

$$a_n = \frac{n}{1+\sqrt{n}}. \quad s_1 = a_1 = \frac{1}{1+\sqrt{1}} = 0.5$$

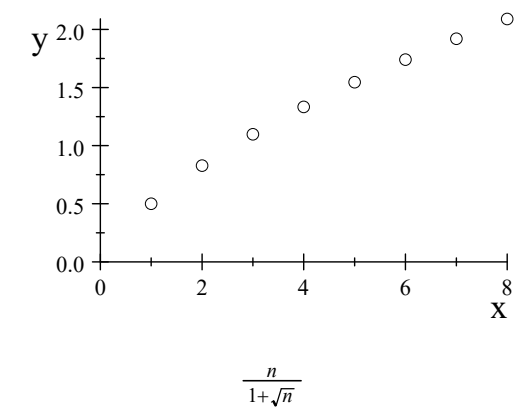
$$s_2 = s_1 + a_2 = 0.5 + \frac{2}{1+\sqrt{2}} \approx 1.3284.$$

$$s_3 = s_2 + a_3 \approx 2.4265$$

$$s_4 \approx 3.7598 \qquad s_5 \approx 5.3049 \qquad s_6 \approx 7.0443$$

$$s_7 \approx 8.9644 \qquad s_8 \approx 11.0540.$$

It appears that the series is divergent (and it is!).



Problem #24 Determine whether the geometric series $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$ is convergent or divergent. If it is convergent, find its sum.

This appears to be a geometric series with ratio $r = \frac{1}{\sqrt{2}}$.

Since $|r| = \frac{1}{\sqrt{2}} < 1$, the series converges.

$$\text{Its sum is } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$$

$$= \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2} (\sqrt{2} + 1) = 2 + \sqrt{2} \approx 3.4142.$$

Problem #40 Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ is convergent or divergent. If it is convergent, find its sum.

To start, note that the series $\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (If it converged, then $\frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} \frac{1}{n}$ would also converge by the Linearity of Convergent Series, but we know from above that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.).

Let's erroneously assume that $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ converges, and see what horrible results follow. Well, since the difference of two convergent series is convergent (again by linearity), then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must also converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent geometric series) and be equal to $\sum_{n=1}^{\infty} \frac{2}{n}$.

But above we saw that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges, so we have a contradiction to our earlier assumption (so it must be wrong), and have instead shown that $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ must diverge.

Problem #48 Determine whether the series $\sum_{i=2}^{\infty} \frac{1}{i^3-i}$ is convergent or divergent by expressing s_n as a telescopic sum. If it is convergent, find its sum.

Using partial fractions, the partial sums of the series $\sum_{i=2}^{\infty} \frac{1}{i^3-i}$ are

$$\begin{aligned} s_n &= \sum_{i=2}^n \frac{1}{i(i-1)(i+1)} = \sum_{i=2}^n \left(-\frac{1}{i} + \frac{\frac{1}{2}}{i-1} + \frac{\frac{1}{2}}{i+1} \right) = \frac{1}{2} \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1} \right) \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + \dots \right. \\ &\quad \left. + \left(\frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \right) + \left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \right) \right] \end{aligned}$$

The middle terms in each set of parentheses cancels with terms located in the adjacent sets of parentheses.

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} \right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2}.$$

$$\text{Thus, } \sum_{n=2}^{\infty} \frac{1}{n^3-n} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2} \right) = \frac{1}{4}.$$

Problem #52 Express the number $0.\overline{46} = 0.46464646\dots$ as a ratio of integers.

$$0.\overline{46} = \frac{46}{100} + \frac{46}{100^2} + \dots$$

is a geometric series with $a = \frac{46}{100}$ and $r = \frac{1}{100}$.

$$\sum_{n=1}^{\infty} ar^{n-1}$$

It converges to: $\frac{a}{1-r} = \frac{\frac{46}{100}}{1-\frac{1}{100}} = \frac{46}{99}$.

Problem #62 Find the values of x for which the series $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$ converges. Find the sum of the series for those values of x .

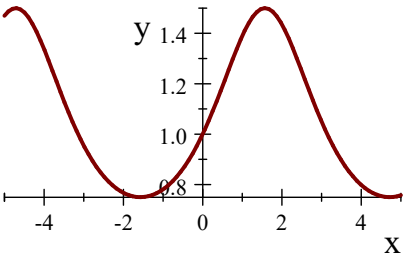
$$\sum_{n=0}^{\infty} \left(\frac{\sin x}{3}\right)^n$$

is a geometric series with $r = \frac{\sin x}{3}$,

so the series converges when $|r| < 1$ or $\left|\frac{\sin x}{3}\right| < 1$

$$\Rightarrow |\sin x| < 3, \text{ which is true for all } x.$$

Thus, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{3}} = \frac{3}{3-\sin x}$.



$$\frac{3}{3-\sin x}$$