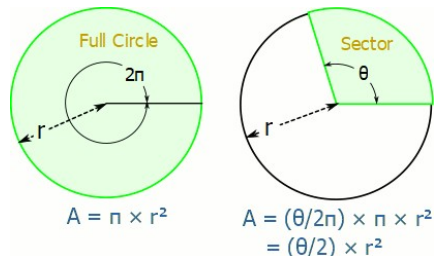


Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

10.4 Areas and Lengths in Polar Coordinates

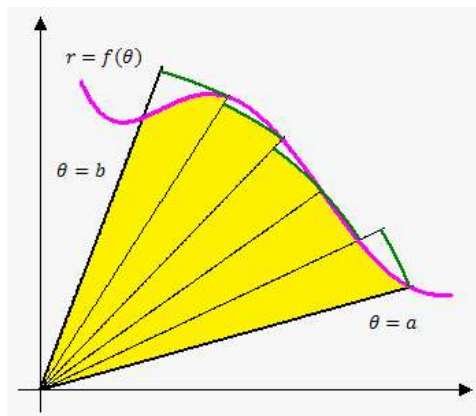
Recall that the area of a sector of a circle is $A = \frac{1}{2}r^2\theta$ where θ is the angle defining the sector (when $\theta = 2\pi$, we get the familiar $A = \pi r^2$).



More generally, we wish to determine the area swept out by the ray connecting the origin $(0,0)$ to a polar curve $r = f(\theta)$ as θ increases from $\theta_0 = a$ to $\theta_1 = b$. We do it similarly to how we estimated the area under the curve using a Riemann approximation. We divide up the interval $[a, b]$ into n segments. Then, adjusting our area formula above, we have:

♦ Riemann Approximation: $A \approx \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$.

♦ Exact area: $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$.



Note that the above works for positive continuous $f(\theta)$, and intervals $[a, b]$ that are less than 2π . Determining the area for situations that fall outside these criteria is similarly achieved, but just requires a few slight alterations. Think about what these alterations might be!

Arc Length

The polar arc length of $r = f(\theta)$ over $a \leq \theta \leq b$ is determined by altering the arc length rule from 10.2. Find the exact derivation in your text, but the result is:

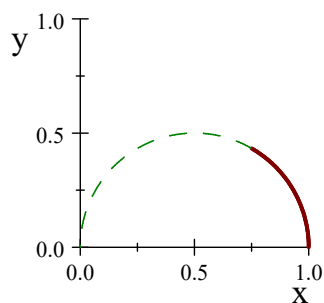
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_a^b \sqrt{f^2(\theta) + \left(\frac{df}{d\theta}\right)^2} d\theta.$$

Problem #2 Find the area of the region that is bounded by the curve $r = \cos\theta$ and also lies in the sector $0 \leq \theta \leq \frac{\pi}{6}$.

$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 d\theta$$

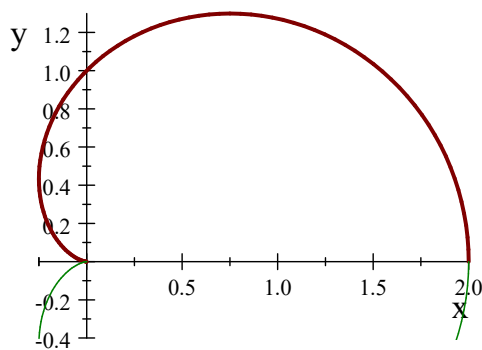
$$= \frac{1}{2} \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{6}} \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}} = \frac{1}{4} \left(\frac{\pi}{6} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) = \frac{\pi}{24} + \frac{\sqrt{3}}{16}.$$



$$r = \cos \theta$$

Problem #6 Find the area of the region defined by the curve $r = 1 + \cos \theta$, with $0 \leq \theta \leq \pi$ (called a cardioid, because of its resemblance to a heart).



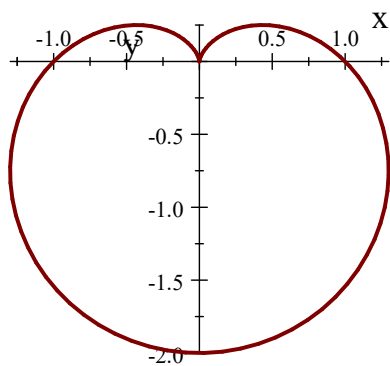
$$r = 1 + \cos \theta$$

$$A = \int_0^{\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left[1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta) \right] d\theta = \frac{1}{2} \int_0^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{1}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi} = \frac{1}{2} \left(\frac{3}{2} \pi + 0 + 0 \right) - \frac{1}{2} (0) = \frac{3}{4} \pi.$$

Problem #10 Sketch the curve $r = 1 - \sin \theta$, and find the area that it encloses.



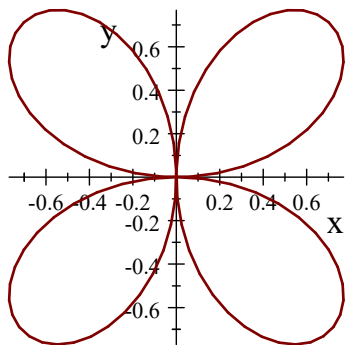
$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (1 - \sin\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta) \right] d\theta = \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= \frac{1}{2} \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{1}{2} [(3\pi + 2) - 2] = \frac{3}{2}\pi.$$

Problem #18 Find the area of the region enclosed by one loop of the curve $r^2 = \sin 2\theta$.



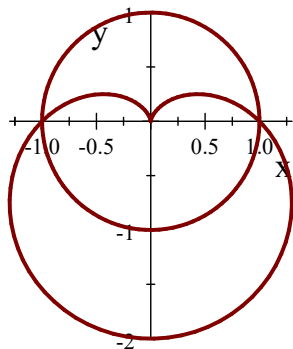
For $\theta = 0$ to $\theta = \frac{\pi}{2}$ the loop in the first quadrant is traced out by $r = \sqrt{\sin 2\theta}$.

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\theta d\theta$$

$$= \left[-\frac{1}{4} \cos 2\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{4} - \left(-\frac{1}{4} \right) = \frac{1}{2}.$$

Problem #24 Find the area of the region that lies inside the curve $r = 1 - \sin\theta$ and outside the curve $r = 1$.



Where do we start and end our integration?

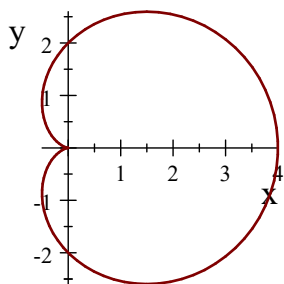
To find the θ values where these curves intersect, we set the equations equal to each other:

$$1 - \sin \theta = 1 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, \text{ etc.}$$

Observe $1 - \sin \theta$ reaches 2 when $\sin \theta = -1$ (between π and 2π).

$$\begin{aligned} \text{So, } A &= \int_{\pi}^{2\pi} \frac{1}{2} [(1 - \sin \theta)^2 - 1^2] d\theta = \frac{1}{2} \int_{\pi}^{2\pi} (\sin^2 \theta - 2 \sin \theta) d\theta \\ &= \frac{1}{4} \int_{\pi}^{2\pi} (1 - \cos 2\theta - 4 \sin \theta) d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta + 4 \cos \theta \right]_{\pi}^{2\pi} \\ &= \frac{1}{4} (2\pi - 0 + 4) - \frac{1}{4} (\pi - 0 - 4) = \frac{\pi}{4} + 2. \end{aligned}$$

Problem #48 Find the exact length of the polar curve $r = 2(1 + \cos \theta)$.



$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{[2(1 + \cos \theta)]^2 + (-2 \sin \theta)^2} d\theta = \int_0^{2\pi} \sqrt{4 + 8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8 \cos \theta + 4(\cos^2 \theta + \sin^2 \theta)} d\theta = \int_0^{2\pi} \sqrt{8 + 8 \cos \theta} d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{1 + \cos \theta} d\theta \end{aligned}$$

$$= \sqrt{8} \int_0^{2\pi} \sqrt{2 \cdot \frac{1}{2} (1 + \cos \theta)} d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{2 \cos^2 \frac{\theta}{2}} d\theta$$

$$= \sqrt{8} \sqrt{2} \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta = 4 \cdot 2 \int_0^{\pi} \cos \frac{\theta}{2} d\theta \quad (\text{by symmetry and } \sqrt{8} \sqrt{2} = 4)$$

$$= 8 \left[2 \sin \frac{\theta}{2} \right]_0^{\pi} = 8 \left(2 \sin \frac{\pi}{2} - 0 \right) = 8(2) = 16.$$