

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

10.2 Calculus with Parametric Curves

Let's say we wish to calculate the slope $\frac{dy}{dx}$, where $x = f(t)$ and $y = g(t)$ are differentiable functions of t . Additionally, assume y is a differentiable function of x . Since we can differentiate f and g with respect to t , we have $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Now, observe that with the chain rule we have $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. Therefore, assuming $\frac{dx}{dt} \neq 0$, we can divide the previous equation by $\frac{dx}{dt}$ to get a **slope equation**: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Area Under the Curve

Recall that the area under the curve for $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$. If a curve is traced out by the parametric equations $x = f(t)$ and $y = g(t)$, on $\alpha \leq t \leq \beta$, the area under the curve (using the substitution rule for definite integrals) is: $A = \int_a^b ydx = \int_a^b g(t)d[f(t)] = \int_\alpha^\beta g(t)f'(t)dt$, where $\beta = f^{-1}(b)$ and $\alpha = f^{-1}(a)$.

Arc Length

Expanding from our definition of arc length from section 8.1, using the substitution rule again with our parameterization, assuming $\frac{dx}{dt} \neq 0$, we have $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \frac{dx}{dt} dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$, where $\beta = f^{-1}(b)$ and $\alpha = f^{-1}(a)$.

Surface Area of Rotated Curve

Similarly, adapting the formula from section 8.2 for surface area of our parametric curve rotated about the x -axis, we find the following formula: $S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$, where $\beta = f^{-1}(b)$ and $\alpha = f^{-1}(a)$.

Problem #2 Find $\frac{dy}{dx}$ for $x = \frac{1}{t}$ and $y = \sqrt{t}e^{-t}$.

$$\begin{aligned}\frac{dy}{dt} &= t^{\frac{1}{2}}(-e^{-t}) + e^{-t}\left(\frac{1}{2}t^{-\frac{1}{2}}\right) \\ &= \frac{1}{2}t^{-\frac{1}{2}}e^{-t}(-2t + 1) = \frac{-2t+1}{2t^{\frac{1}{2}}e^t},\end{aligned}$$

$$\text{and } \frac{dx}{dt} = -\frac{1}{t^2}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{-2t+1}{2t^{\frac{1}{2}}e^t} \left(-\frac{t^2}{1}\right) = \frac{(2t-1)t^{\frac{3}{2}}}{2e^t}.\end{aligned}$$

Problem #4 Find an equation of the tangent to the curve $x = t - t^{-1}$, $y = 1 + t^2$, at the point corresponding to the value of the parameter $t = 1$.

$$\frac{dy}{dt} = 2t,$$

$$\frac{dx}{dt} = 1 + t^{-2} = \frac{t^2 + 1}{t^2},$$

$$\text{and } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2t \left(\frac{t^2}{t^2 + 1} \right) = \frac{2t^3}{t^2 + 1}.$$

When $t = 1$, $(x, y) = (0, 2)$

$$\text{and } \frac{dy}{dx} = \frac{2}{2} = 1.$$

So an equation of the tangent to the curve at the point corresponding to $t = 1$ is $y - 2 = 1(x - 0)$, or $y = x + 2$.

Problem #8 Find an equation of the tangent to the curve $x = 1 + \sqrt{t}$, $y = e^{t^2}$ at the point $(2, e)$ by two methods: **a) without eliminating the parameter.**

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = e^{t^2} \cdot 2t,$$

$$\text{and } \frac{dy}{dx} = 2te^{t^2} (2\sqrt{t}) = 4t^{\frac{3}{2}} e^{t^2}.$$

So we have the slope of the tangent line for any t , now what do we need?

$$\text{At } (2, e), \text{ we have } 2 = 1 + \sqrt{t} \Rightarrow \sqrt{t} = 1 \Rightarrow t = 1,$$

$$\text{so } \frac{dy}{dx} = 4e.$$

$$\text{Finally, } y - e = 4e(x - 2) \Rightarrow y = 4ex - 7e.$$

b) Find the equation by first eliminating the parameter.

$$\sqrt{t} = x - 1 \Rightarrow t = (x - 1)^2$$

(since t was square rooted, we assume it is positive since we are dealing with real numbers)

$$\text{So } y = e^{t^2} = e^{(x-1)^4}.$$

To determine the line, we need a slope:

$$y' = e^{(x-1)^4} \cdot 4(x-1)^3.$$

At $(2, e)$, we have $y' = e \cdot 4 = 4e$.

So an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$ (same as before).

Problem #14 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = t^2 + 1$, $y = e^t - 1$. For which values of t is the curve concave upward?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t}{2t}$$

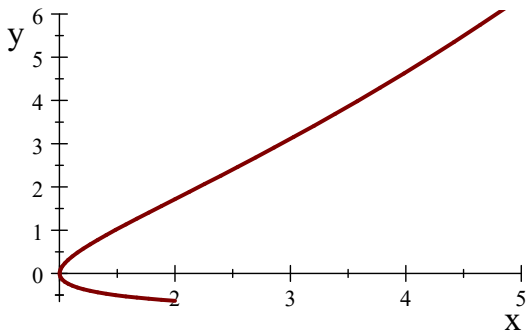
But what is $\frac{d^2y}{dx^2}$? It definitely is NOT $\left(\frac{dy}{dx}\right)^2$!! Nor is it $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. You should understand it theoretically to be $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. But how do we take the derivative with respect to x of a function in t ? The key is recognizing that $\frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}$. Using this, we have:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}\left(\frac{dy}{dx}\right) \\ &= \frac{1}{2t} \frac{2te^t - e^t \cdot 2}{(2t)^2} = \frac{2e^t(t-1)}{(2t)^3} = \frac{e^t(t-1)}{4t^3}. \end{aligned} \quad (\text{Are we done?})$$

The curve is concave up when $\frac{d^2y}{dx^2} > 0$,

Observe that the numerator is negative when $t < 1$, and the denominator when $t < 0$.

So, $\frac{d^2y}{dx^2} > 0$ when $t < 0$ and $1 < t$.



$$(x, y) = (t^2 + 1, e^t - 1)$$

Problem #18 Find the points on the curve $x = t^3 - 3t$, $y = t^3 - 3t^2$ where the tangent is horizontal or vertical. If you have a

graphing device, graph the curve to check your work.

$\frac{dy}{dt} = 3t^2 - 6t = 3t(t - 2)$, and $\frac{dx}{dt} = 3t^2 - 3 = 3(t + 1)(t - 1)$.

So $\frac{dy}{dt} = 0$ when $t = 0$ or 2 .

$\Rightarrow (x,y) = (0,0)$ or $(2,-4)$.

Also $\frac{dx}{dt} = 0$ when $t = -1$ or 1 .

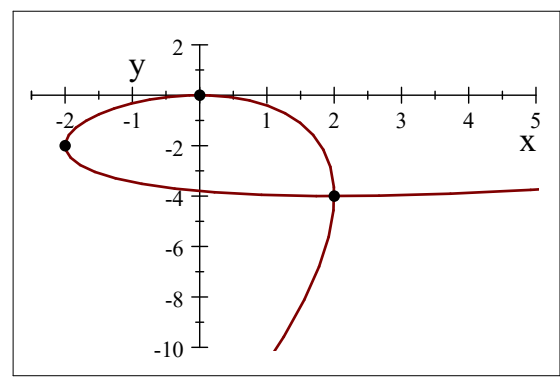
$\Rightarrow (x,y) = (2,-4)$ or $(-2,-2)$.

The curve has horizontal tangents at $(0,0)$ and $(2,-4)$, and vertical tangents at $(2,-4)$ and $(-2,-2)$.

So, the curve has both horizontal and vertical tangents at $(2,-4)$!!?



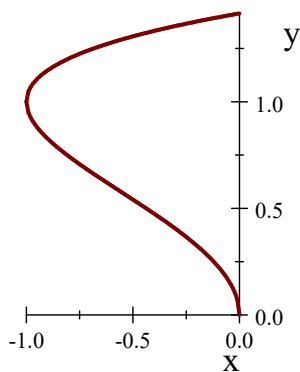
See the graph below.



Problem #32 Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

The curve intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$.

The corresponding values of y are 0 and $\sqrt{2} \approx 1.414$.



$$x = t^2 - 2t, \quad y = \sqrt{t}$$

The area enclosed by the curve and the y-axis is given by: $\int_{y=0}^{y=\sqrt{2}} (x_R - x_L) dy$

$$= \int_{y=0}^{y=\sqrt{2}} (0 - x(y)) dy$$

$$= \int_{t=0}^{t=2} [0 - x(t)] y'(t) dt$$

$$= - \int_0^2 (t^2 - 2t) \left(\frac{1}{2\sqrt{t}} \right) dt$$

$$= - \int_0^2 \left(\frac{1}{2} t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) dt = - \left[\frac{1}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right]_0^2 = - \left(\frac{1}{5} \cdot 2^{\frac{5}{2}} - \frac{2}{3} \cdot 2^{\frac{3}{2}} \right)$$

$$= -2^{\frac{1}{2}} \left(\frac{2^2}{5} - \frac{2^2}{3} \right) = -\sqrt{2} \left(-\frac{8}{15} \right) = \frac{8\sqrt{2}}{15}.$$

Problem #42 Find the exact length of the curve $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$.

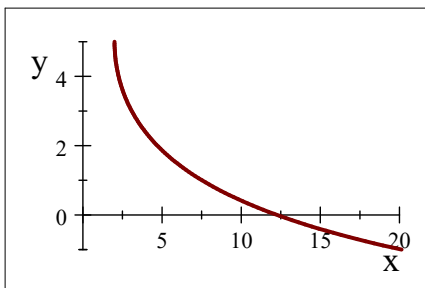
$$\frac{dx}{dt} = e^t - e^{-t} \text{ and } \frac{dy}{dt} = -2.$$

$$\text{So } \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = (e^{2t} - 2 + e^{-2t}) + 4 = e^{2t} + 2 + e^{-2t}.$$

$$\text{Thus, } L = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}} dt$$

$$\text{Observe that: } e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2.$$

$$\text{Thus, } L = \int_0^3 e^t + e^{-t} dt = [e^t - e^{-t}]_0^3 = (e^3 - e^{-3}) - (1 - 1) = e^3 - e^{-3} \approx 20.$$



$$x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \leq t \leq 3$$

Problem #52 Find the distance traveled by a particle with position $(x, y) = (\cos^2 t, \cos t)$ as t varies in the time interval $0 \leq t \leq 4\pi$.

$$(\text{Distance Traveled}) = \int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2 \cos t \sin t)^2 + (-\sin t)^2 = \sin^2 t (4 \cos^2 t + 1).$$

$$(\text{Distance Traveled}) = \int_0^{4\pi} |\sin t| \sqrt{4 \cos^2 t + 1} dt = 4 \int_0^{\pi} \sin t \sqrt{4 \cos^2 t + 1} dt$$

$$= -4 \int_1^{-1} \sqrt{4u^2 + 1} du \quad (\text{where } u = \cos t, \quad du = -\sin t dt)$$

$$= 4 \int_{-1}^1 \sqrt{4u^2 + 1} du = 8 \int_{u=0}^{u=1} \sqrt{4u^2 + 1} du$$

Let $2u = \tan \theta$ and $2du = \sec^2 \theta d\theta$. Then:

$$(\text{Distance Traveled}) = 8 \int_{\frac{\tan \theta}{2}=0}^{\frac{\tan \theta}{2}=1} \sqrt{\tan^2 \theta + 1} \left(\frac{\sec^2 \theta d\theta}{2} \right) = 4 \int_{\theta=0}^{\theta=\tan^{-1}2} \sec^2 \theta \sqrt{\sec^2 \theta} d\theta$$

$$= 4 \int_0^{\tan^{-1}2} \sec^3 \theta d\theta.$$

The solution to this integral is given in the list in the back of the book by Equation 71:

$$= 2[\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|]_0^{\tan^{-1}2}$$

$$= 2(\sec(\tan^{-1}2) \tan(\tan^{-1}2) + \ln|\sec(\tan^{-1}2) + \tan(\tan^{-1}2)|) - 2(\sec 0 \tan 0 + \ln|\sec 0 + \tan 0|)$$

$$= 2(2 \sec(\tan^{-1}2) + \ln|\sec(\tan^{-1}2) + 2|) - 2(0 + \ln 1)$$

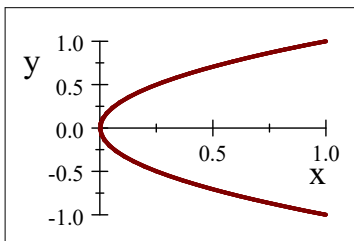
We wish to calculate $\sec t$, where $t = \tan^{-1}2$.

$2 = \frac{\text{opposite}}{\text{adjacent}}$. So we have that opposite = 2 and adjacent = 1.

Therefore, we have a right triangle in which hypotenuse = $\sqrt{2^2 + 1^2} = \sqrt{5}$.

So we can calculate $\sec(\tan^{-1} 2) = \sec(t) = \frac{1}{\cos t} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{5}}{1} = \sqrt{5}$.

Therefore, we have: (Distance Traveled) = $2(2\sqrt{5} + 2\ln(\sqrt{5} + 2)) - 0$
 $= 4\sqrt{5} + 2\ln(\sqrt{5} + 2) \approx 11.83$.



$$(x, y) = (\cos^2 t, \cos t)$$

Problem #62 Find the exact area of the surface obtained by rotating the curve $x = 3t - t^3$, $y = 3t^2$, on $0 \leq t \leq 1$ about the x-axis.

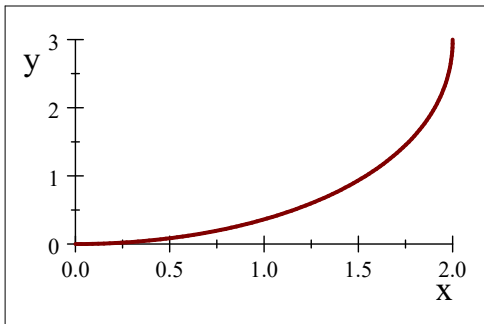
Recall: $S = \int_{t=a}^{t=\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$.

$$\text{So, } \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (9t^4 - 18t^2 + 9) + 36t^2 = 9(t^4 + 2t^2 + 1).$$

$$\text{And, } S = \int_0^1 18\pi t^2 \sqrt{t^4 + 2t^2 + 1} dt.$$

But observe that: $t^4 + 2t^2 + 1 = (t^2 + 1)^2$. Therefore,

$$S = \int_0^1 18\pi t^2 (t^2 + 1) dt = 18\pi \int_0^1 t^4 + t^2 dt = 18\pi \left[\frac{1}{5} t^5 + \frac{1}{3} t^3 \right]_0^1 = \frac{48}{5} \pi.$$



$$x = 3t - t^3, \quad y = 3t^2, \quad \text{on } 0 \leq t \leq 1$$

