

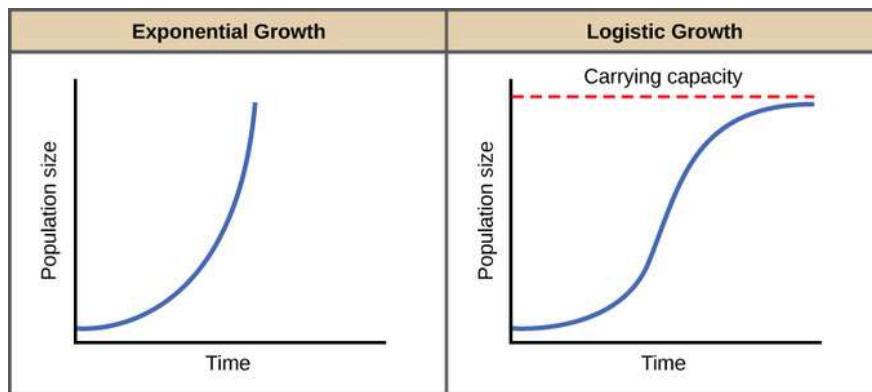
Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

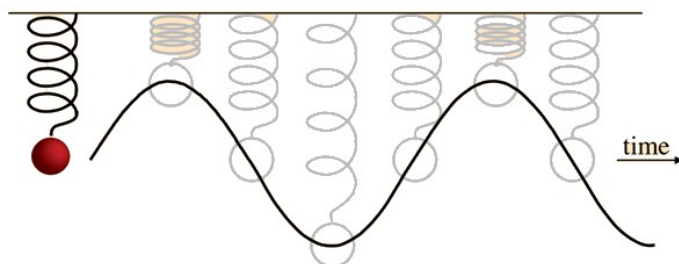
9.1 - Modeling with Differential Equations (DEQ)

Models of Population Growth: If we label $P(t)$ the population at some time t , then the **first-order differential equations** (DEQ) $\frac{dP}{dt} = kP$ or $P' = kP$ represent the rate of change in the population being proportional to the population itself, where k is some constant.

Logistic Differential Equation: $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ represents a more complex and realistic model for population growth which recognizes that a **limiting population** or **carrying capacity** (M) affects the population curve.



Model for the Motion of a Spring:



For a spring fixed at one end, and loose at the other end, we define the position of the loose end as $x(t)$. We also define the **equilibrium position** as $x = 0$, where the spring exerts no force on the loose end, and we define the **Restoring Force** $= -kx$, where k is a constant representing the so-called “**spring constant**,” which is the strength with which the spring accelerates the unfixed end to “restore” the spring back to the equilibrium $x = 0$.

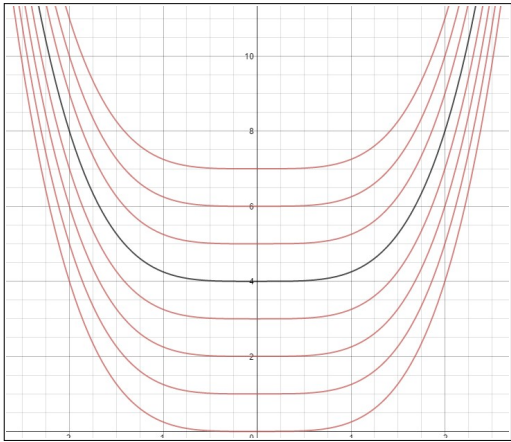
If we ignore friction/resistance, then since we know that “force equals mass times acceleration” or $F = ma$, we get $-kx = ma = m \frac{d^2x}{dt^2} = x''$ or rearranging: $x'' = -\frac{k}{m}x$, a **second-order differential equation**.

Differential Equations (like $x'' = -\frac{k}{m}x$) are equations where our task as mathematicians is not to solve for a variable x as usual, but rather we are seeking to find a function $(x(t))$. In the above example, our solution to the second order differential equation would be a function $x(t)$ which would (due to the periodic motion of a spring), involve trigonometric terms; for example one solution is $x(t) = 7 \sin(3t)$.

Simple Example:

For the differential equation $y' = x^3$, we know how to integrate this to discover $y = \frac{x^4}{4} + C$. Below, you see the **family of solutions** for this differential equation. Each solution visualized below is the result of our choice of C . For each value of C , you

have a **particular solution** to the differential equation.



Initial Value Problems (IVP):

If in the above example, we were told that $y(2) = 8$, then having this extra bit of information means we can solve for C . Plugging $y = 8$ and $t = 2$ into our equation, we get:

$$8 = \frac{(2)^4}{4} + C = \frac{16}{4} + C = 4 + C, \quad \Rightarrow \quad C = 4.$$

Therefore, $y = \frac{x^4}{4} + 4$ is the particular solution for this IVP.

Problem #2 Verify that $y = -t \cos t - t$ is a solution of the initial value problem: $t \frac{dy}{dt} = y + t^2 \sin t$, $y(\pi) = 0$.

$$\frac{dy}{dt} = (-1) \cos t - t(-\sin t) - 1 = t \sin t - \cos t - 1.$$

So the left hand side of the differential equation (DEQ) is: $t \frac{dy}{dt} = t(t \sin t - \cos t - 1) = t^2 \sin t + (-t \cos t - t)$.

$$= t^2 \sin t + y \text{ which is the right-hand side of the DEQ.}$$

So y IS a solution of the general differential equation. (Are we done?)

Also, $y(\pi) = -\pi \cos \pi - \pi = -\pi(-1) - \pi = \pi - \pi = 0$, so the initial condition is satisfied and our solution solves the IVP.

Problem #10 A function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$.

a) What are the constant solutions of the equation?

Observe that if $y = k$ is a constant, then $y' = 0$.

$$\text{So, } \frac{dy}{dt} = y^4 - 6y^3 + 5y^2 = k^4 - 6k^3 + 5k^2 = 0$$

$$k^2(k^2 - 6k + 5) = 0$$

$$k^2(k-1)(k-5) = 0 \Leftrightarrow k = 0, 1, \text{ or } 5.$$

(solutions for $\frac{dy}{dt} = 0$ also called equilibrium points)

b) For what values of y is y increasing?

$$y \text{ is increasing implies that } \frac{dy}{dt} = y^2(y-1)(y-5) > 0.$$

Plugging in values of y in between our equilibrium points, tells us whether those intervals are increasing. So we test:

$y = -1$, $y = \frac{1}{2}$, $y = 2$, and $y = 6$. As a shortcut, don't calculate the actual value of $\frac{dy}{dt}$, but just keep track of the sign of the factors.

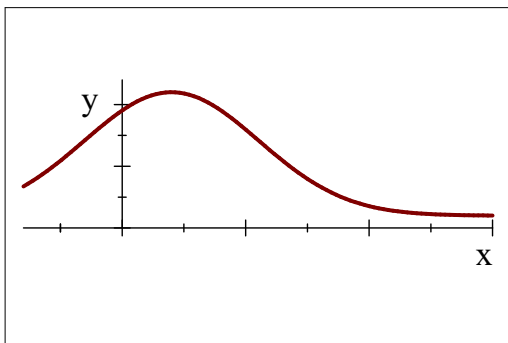
$$y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty).$$

c) For what values of y is y decreasing?

$$y \text{ is decreasing implies that } \frac{dy}{dt} = y^2(y-1)(y-5) < 0.$$

$$y \in (1, 5).$$

Problem #12 The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.



a) $y' = 1 + xy$ b) $y' = -2xy$, c) $y' = 1 - 2xy$.

a) $y' = 1 + xy$,

Observe that $y' = 1 + xy > 1$ for points in the first quadrant,
but we can see that $y' < 0$ for some points in the first quadrant.

b) $y' = -2xy$,

$$y' = 1 - 2xy = 0, \text{ when } x = 0. \text{ But we can see that } y' > 0 \text{ for } x = 0.$$

c) $y' = 1 - 2xy$.

♦ When $x = 0$, y' could reasonably be 1.

♦ When $x < 0$, y' could be greater than 1.

♦ Solving $y' = 1 - 2xy$ for y gives us $y = \frac{1-y'}{2x}$.

If y' takes on small negative values, then as $x \rightarrow \infty$, $y \rightarrow 0^+$, as shown in the figure.

So it appears that c is a good candidate is our solution.