

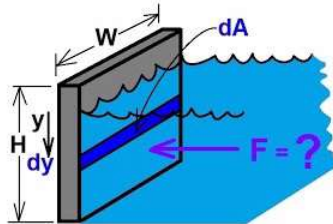
Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

Will not cover the center of mass of a region lying between two curves $f(x)$ and $g(x)$ (Eq. 9 on page 564), nor will we cover the Theorem of Pappus on page 565.

We will cover **center-of-mass** for regions of the form $0 < y < f(x)$ and $a < x < b$ discussed earlier in the section.

8.3 - Applications to Physics and Engineering



Hydrostatic Pressure and Force:

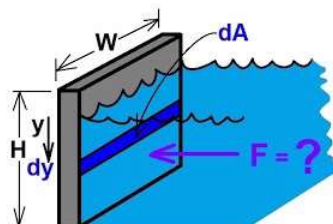
Definition: **Hydrostatics** is the branch of fluid mechanics that studies fluids at rest. It encompasses the study of the conditions under which fluids are at rest in stable equilibrium as opposed to fluid dynamics (the study of fluids in motion).

Let:

- ♦ F be the **hydrostatic force**
(the force felt on a surface within a fluid due to the pressure of that fluid at rest).
- ♦ P be the **hydrostatic pressure** (the force at a particular point on the surface),
- ♦ g be the force of **gravity**,
- ♦ ρ be the **density** of the fluid,
- ♦ A be the **surface area** of a plate,
- ♦ d be the **depth** of the plate below the surface,
- ♦ V be the **volume** of the fluid above the plate, and
- ♦ $m = \rho V = \rho dA$ be the **mass** of the fluid above the plate.

Then, the **force** on the plate can be written as:

- ♦ $F = mg = \rho g dA$, where the **hydrostatic pressure** is:
- ♦ $P = \frac{F}{A} = \rho g d$ (point force/pressure)



For water, $\rho = 1000\text{ kg/m}^3$ or 0.0624 lb/ft^3 .

The hydrostatic force on a dam where W is the width of the dam, the water is H meters high, and where height is measured on a vertical y -axis is:

$$F = \int_{d=0}^H \rho g d A = \int_{y=0}^H 1000(9.8)y(Wdy).$$

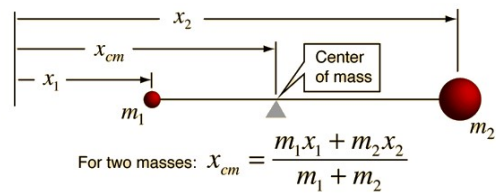
Moments and Centers of Mass:

Moments of an n -particle system about the x -axis and y -axis.

$$M_x = \sum_{i=1}^n m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^n m_i x_i, \text{ for } n \text{ particles with masses } m_i.$$

Center of Mass in 1D is $\bar{x}$ where $\bar{x} = \frac{m_1 x_1 + \dots + m_n x_n}{m}$

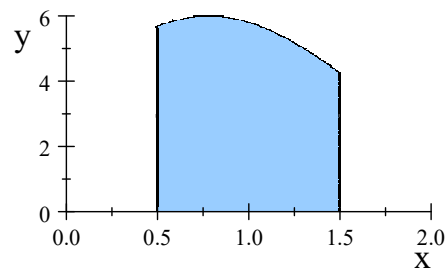
and $m = \sum m_i$, (the total mass).



The **Center of Mass in 2D** is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{m}$, and $\bar{y} = \frac{M_x}{m}$.

For a flat plate with uniform density ρ , the center of mass is called the **centroid**. The **symmetry principle** says that if a flat plate is symmetric about a line, then the centroid is located on that line.

Imagine a flat plate (with constant density ρ) sitting on the x -axis where the shape of the top of the plate is defined by $f(x)$ (see the figure). How do we figure out its moments and center of mass?



First, divide it into n vertical strips where the edges of these strips are located at $\{x_0, x_1, \dots, x_n\}$. Also, let $\bar{x}_i$ be the midpoint between x_{i-1} and x_i . For example, $\bar{x}_1 = \frac{x_0 + x_1}{2}$.

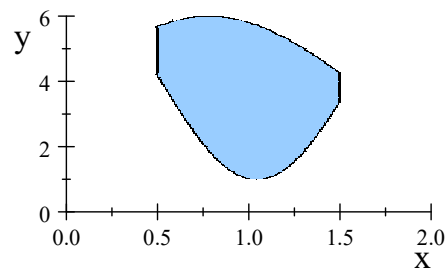
The **surface area** of the shape is $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\bar{x}_i) \Delta x = \int_{x=a}^b f(x) dx$. And therefore the **mass** of the plate is $m = \rho A = \rho \int_{x=a}^b f(x) dx$.

The **moment about the y-axis** of the flat plate is: $M_y = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho \bar{x}_i f(\bar{x}_i) \Delta x = \rho \int_{x=a}^b x f(x) dx$.

And the **moment about the x-axis**: $M_x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{\rho}{2} [f(\bar{x}_i)]^2 \Delta x \right] = \frac{\rho}{2} \int_{x=a}^b [f(x_i)]^2 dx$.

Therefore, the **center of mass** of the above plate is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{m} = \frac{1}{A} \int_a^b xf(x)dx$, and $\bar{y} = \frac{M_x}{m} = \frac{1}{2A} \int_a^b [f(x)]^2 dx$. (observe how the ρ 's canceled out in the fractions).

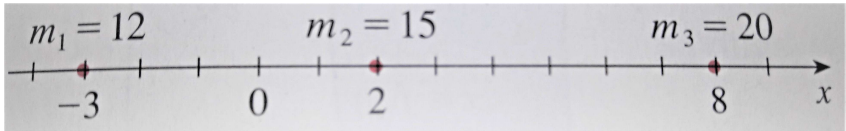
The rest of the review is optional:



In the image above, let $f(x)$ be the upper curve, and $g(x)$ be the lower curve of the plate.

The **center of mass** is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)]dx$, and $\bar{y} = \frac{1}{2A} \int_a^b \{[f(x)]^2 - [g(x)]^2\}dx$.

Theorem of Pappus: If you have a flat plate that is rotated about a line that is to one side of the plate, then the volume of the solid-of-revolution is $A \cdot d$ where A is the area of the plate, and d is the distance traveled by the centroid of the plate.



Problem #22 Point masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\bar{x}$.

The moment M is $m_1x_1 + m_2x_2 + m_3x_3 = 12(-3) + 15(2) + 20(8) = 154$.

The mass m is $m_1 + m_2 + m_3 = 12 + 15 + 20 = 47$.

The center of mass is $\bar{x} = \frac{M}{m} = \frac{154}{47} \approx 3.2766$.

Problem #23 The masses $m_1 = 4$, $m_2 = 2$, $m_3 = 4$ are located at the points $P_1(2, -3)$, $P_2(-3, 1)$, $P_3(3, 5)$. Find the moments M_x and M_y and the center of mass of the system.

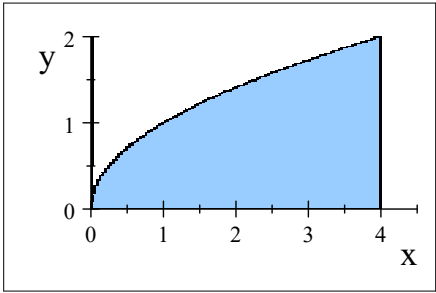
The mass is $m = \sum_{i=1}^3 m_i = 4 + 2 + 4 = 10$.

The moment about the x-axis is $M_x = \sum_{i=1}^3 m_iy_i = 4(-3) + 2(1) + 4(5) = 10$.

The moment about the y-axis is $M_y = \sum_{i=1}^3 m_ix_i = 4(2) + 2(-3) + 4(3) = 14$.

The center of mass is $(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}) = (\frac{14}{10}, \frac{10}{10}) = (1.4, 1)$.

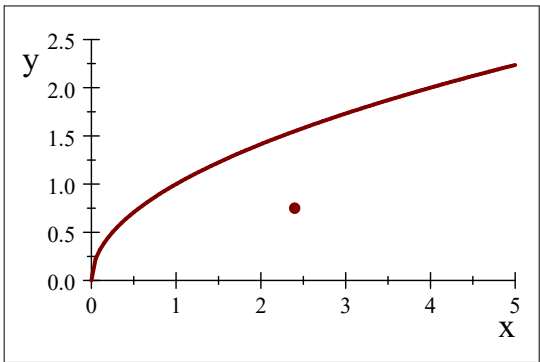
Problem #26 Sketch the region bounded by the curves $y = \sqrt{x}$, $y = 0$, $x = 4$, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.



The region and the figure is “right-heavy” and “bottom-heavy,” so we know that $\bar{x} > 2$ and $\bar{y} < 1$, and we might guess that $\bar{x} \approx 2.3$ and $\bar{y} \approx 0.8$.

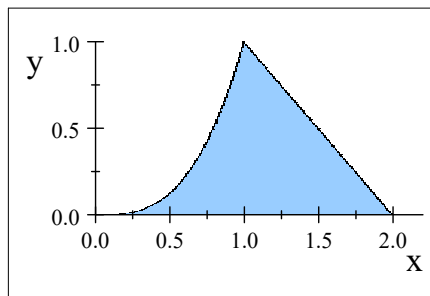
$$A = \int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^4 = \frac{16}{3}.$$
$$\bar{x} = \frac{1}{A} \int_0^4 x(\sqrt{x}) \, dx = \frac{3}{16} \int_0^4 x^{\frac{3}{2}} \, dx = \frac{3}{16} \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = \frac{3}{40} (32 - 0) = \frac{12}{5}.$$
$$\bar{y} = \frac{1}{A} \int_0^4 \frac{1}{2} (\sqrt{x})^2 \, dx = \frac{3}{16} \int_0^4 \frac{1}{2} x \, dx = \frac{3}{32} \left[\frac{1}{2} x^2 \right]_0^4 = \frac{3}{64} (16 - 0) = \frac{3}{4}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4} \right)$.



Centroid

Problem #32 Find the centroid of the region bounded by the curves $y = x^3$, $x + y = 2$, $y = 0$.



First we will want to find the area by integrating under the curve. However,our curve is determined by two functions. Where does one function end, and the other function start?

$x^3 = 2 - x \Rightarrow x^3 + x - 2 = 0$, we could attempt to solve this cubic, or we can observe from the graph that the two functions appear to meet at $x = 1$, and indeed this is a solution to the cubic.

$$A = \int_0^1 x^3 dx + \int_1^2 (2 - x) dx = \left[\frac{1}{4} x^4 \right]_0^1 + \left[2x - \frac{1}{2} x^2 \right]_1^2$$

$$= \frac{1}{4} + (4 - 2) - \left(2 - \frac{1}{2} \right) = \frac{3}{4}.$$

$$\bar{x} = \frac{1}{A} \left[\int_0^1 x(x^3) dx + \int_1^2 x(2 - x) dx \right] = \frac{4}{3} \left[\int_0^1 x^4 dx + \int_1^2 (2x - x^2) dx \right]$$

$$= \frac{4}{3} \left\{ \left[\frac{1}{5} x^5 \right]_0^1 + \left[x^2 - \frac{1}{3} x^3 \right]_1^2 \right\} = \frac{4}{3} \left[\frac{1}{5} + \left(4 - \frac{8}{3} \right) - \left(1 - \frac{1}{3} \right) \right]$$

$$= \frac{4}{3} \left(\frac{13}{15} \right) = \frac{52}{45}.$$

$$\bar{y} = \frac{1}{A} \left[\int_0^1 \frac{1}{2} (x^3)^2 dx + \int_1^2 (2 - x)^2 dx \right] = \frac{2}{3} \left[\int_0^1 x^6 dx + \int_1^2 (x - 2)^2 dx \right]$$

$$= \frac{2}{3} \left\{ \left[\frac{1}{7} x^7 \right]_0^1 + \left[\frac{1}{3} (x - 2)^3 \right]_1^2 \right\} = \frac{2}{3} \left(\frac{1}{7} - 0 + 0 + \frac{1}{3} \right) = \frac{2}{3} \left(\frac{10}{21} \right) = \frac{20}{63}.$$

Thus the centroid is $(\bar{x}, \bar{y}) = \left(\frac{52}{45}, \frac{20}{63} \right)$.