

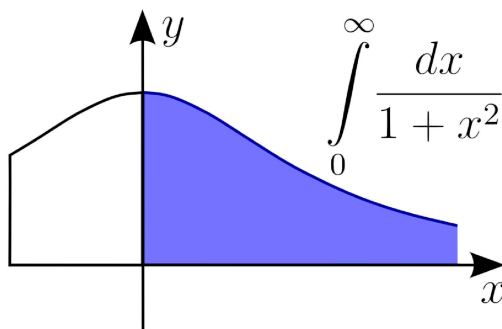
Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.8 - Improper Integrals

Definition: An **improper integral** is the limit of a definite integral $\int_a^b$ as an endpoint (or both endpoints) of the interval of integration approaches either a specified real number (e.g., $b \rightarrow 7$), or positive or negative infinity (e.g., $a \rightarrow -\infty$).

Infinite Intervals [Type 1]



♦ If $\int_a^t f(x)dx$ exists (is finite) for every number $t \geq a$, then ...

$$\int_a^{\infty} f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx, \text{ provided this limit exists (has a limit, and is finite).}$$

♦ If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then ...

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx, \text{ provided this limit exists.}$$

The improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

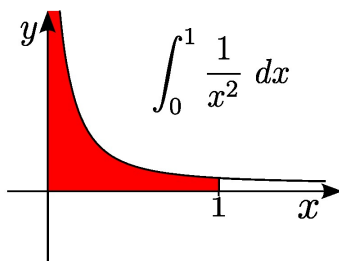
♦ If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx, \text{ for any } a \in \mathbb{R} \text{ (the most convenient choice is often } a = 0 \text{).}$$

The p-test: $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

<https://www.desmos.com/calculator/ejmk1xvet2>

Discontinuous Integrands [Type 2]



♦ If f is continuous on $[a, b)$ and is discontinuous at b , then ...

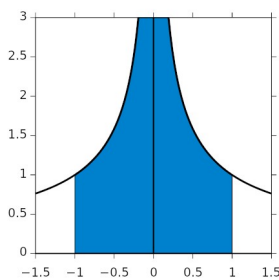
$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx, \text{ if this limit exists (as a finite number).}$$

♦ If f is continuous on $(a, b]$ and is discontinuous at a , then ...

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx \text{ if this limit exists (as a finite number).}$$

Similar to Type I, the improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Internal Discontinuity



♦ If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define $\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$.

Comparison Test for Improper Integrals:

Assume f, g are continuous with $f(x) \geq g(x) \geq 0$ for all $x \geq a$, then:

♦ If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

♦ If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Problem #2 Which of the following integrals are improper? Why?

a) $\int_0^{\frac{\pi}{4}} \tan x dx$, b) $\int_0^\pi \tan x dx$, c) $\int_{-1}^1 \frac{dx}{x^2-x-2}$, d) $\int_0^\infty e^{-x^3} dx$.

a) Since $y = \tan x$ is defined and continuous on $[0, \frac{\pi}{4}]$, $\int_0^{\frac{\pi}{4}} \tan x dx$ is proper.

b) Since $y = \tan x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^\pi \tan x dx$ is a Type 2 improper integral.

c) Since $y = \frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)}$ has an infinite discontinuity at $x = -1$, then $\int_{-1}^1 \frac{dx}{x^2-x-2}$ is a Type 2 improper integral.

d) Since $\int_0^\infty e^{-x^3} dx$ has an infinite interval of integration, it is an improper integral of Type 1.

Problem #40 Determine whether $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$ is convergent or divergent. Evaluate it if it is convergent.

Integrate by parts with $u = \ln x$, $dv = \frac{dx}{\sqrt{x}}$ implies $du = \frac{dx}{x}$, $v = 2\sqrt{x}$.

$$\begin{aligned}\text{So, } \int_0^1 \frac{\ln x}{\sqrt{x}} dx &= [2\sqrt{x} \ln x]_0^1 - 2 \int_0^1 \frac{dx}{\sqrt{x}} \\&= \lim_{t \rightarrow 0^+} \left([2\sqrt{x} \ln x]_t^1 - 2 \int_t^1 \frac{dx}{\sqrt{x}} \right) \\&= \lim_{t \rightarrow 0^+} \left(-2\sqrt{t} \ln t - 4[\sqrt{x}]_t^1 \right) = \lim_{t \rightarrow 0^+} (-2\sqrt{t} \ln t - 4 + 4\sqrt{t}) \\&= -2 \lim_{t \rightarrow 0^+} (\sqrt{t} \ln t) - 4.\end{aligned}$$

Observe that $\lim_{t \rightarrow 0^+} \sqrt{t} \ln t = \lim_{t \rightarrow 0^+} \frac{\ln t}{t^{-\frac{1}{2}}}$

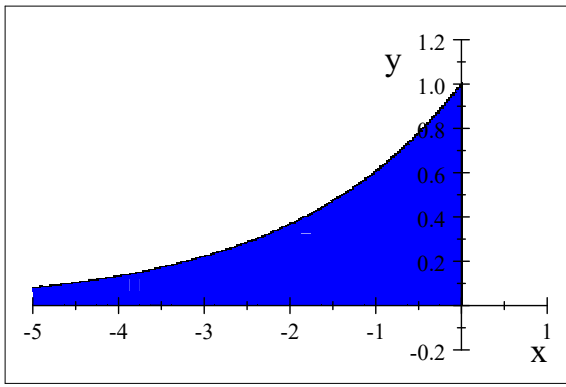
$$\stackrel{L'H}{=} \lim_{t \rightarrow 0^+} \frac{\frac{1}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}} = \lim_{t \rightarrow 0^+} (-2\sqrt{t}) = 0.$$

So $\int_0^1 \frac{\ln x}{\sqrt{x}} dx = -4$ is convergent.

Determine whether $\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}} dx$ is convergent or divergent. Evaluate it if it is convergent.

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}} dx &= \int_{-1}^1 x^{-\frac{2}{3}} dx = \int_{-1}^0 x^{-\frac{2}{3}} dx + \int_0^1 x^{-\frac{2}{3}} dx \\&= \lim_{t \rightarrow 0} \int_{-1}^t x^{-\frac{2}{3}} dx + \lim_{t \rightarrow 0} \int_t^1 x^{-\frac{2}{3}} dx = \lim_{t \rightarrow 0} [3x^{\frac{1}{3}}]_{-1}^t + \lim_{t \rightarrow 0} [3x^{\frac{1}{3}}]_t^1 \\&= \lim_{t \rightarrow 0} \left(3t^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} \right) + \lim_{t \rightarrow 0} \left(3(1)^{\frac{1}{3}} - 3t^{\frac{1}{3}} \right) \\&= (0 + 3) + (3 - 0) = 6 \text{ is convergent.}\end{aligned}$$

Problem #42 Sketch the region $S = \{(x,y) : x \leq 0, 0 \leq y \leq e^x\}$ and find its area (if the area is finite).



$$\text{Area} = \int_{-\infty}^0 e^x dx$$

$$= \lim_{t \rightarrow -\infty} \int_t^0 e^x dx$$

$$= \lim_{t \rightarrow -\infty} [e^x]_t^0 = \lim_{t \rightarrow -\infty} (e^0 - e^t)$$

$$= 1 - 0 = 1.$$



Problem #52 Use the comparison theorem to determine whether $\int_0^\infty \frac{\tan^{-1}x}{2+e^x} dx$ is convergent or divergent.

For $x \geq 0$, we have $0 < \tan^{-1}x < \frac{\pi}{2} < 2$.

$$\text{So } \frac{\tan^{-1}x}{2+e^x} < \frac{2}{2+e^x} \dots$$

$$< \frac{2}{e^x} = 2e^{-x}.$$

$$\text{Now, } I = \int_0^\infty 2e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t 2e^{-x} dx = \lim_{t \rightarrow \infty} [-2e^{-x}]_0^t = \lim_{t \rightarrow \infty} \left(-\frac{2}{e^t} + 2\right) = 2.$$

So I is convergent, and by comparison, $\int_0^\infty \frac{\tan^{-1}x}{2+e^x} dx < I$ is also convergent.

