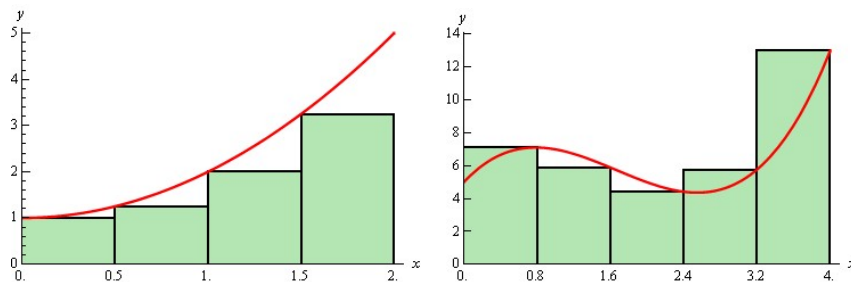


Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.7 - Approximate Integration



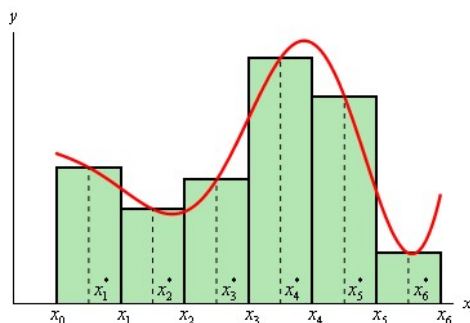
Left and Right Endpoint Approximation:

Recall from Calc I:

$$\int_a^b f(x)dx \approx L_n := \sum_{i=1}^n f(x_{i-1})\Delta x, \text{ and}$$

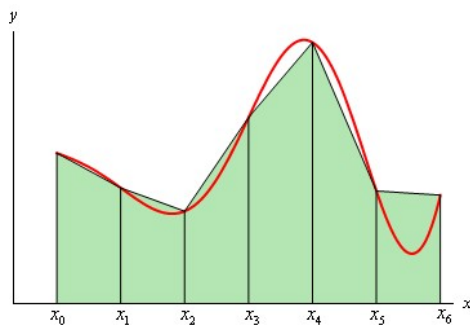
$$\int_a^b f(x)dx \approx R_n := \sum_{i=1}^n f(x_i)\Delta x, \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x.$$

AND,



Midpoint Rule: $\int_a^b f(x)dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)]$, where $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ = midpoint of $[x_{i-1}, x_i]$.

But now we also have:

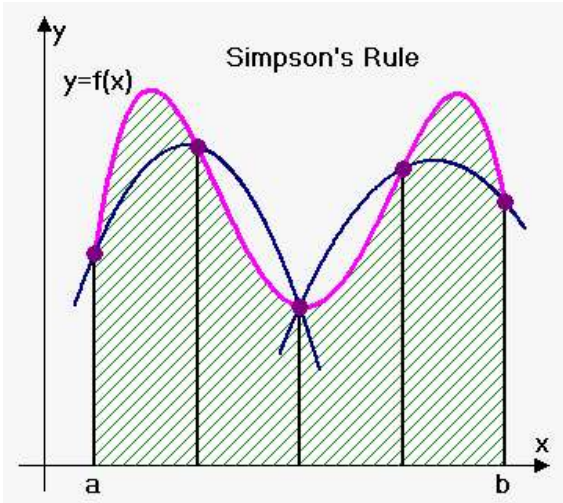


Trapezoidal Rule: (particularly good for periodic functions)

$$\int_a^b f(x)dx \approx T_n := \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)].$$

Error Bounds for Midpoint and Trapezoidal Rules: Suppose $|f''(x)| \leq K$, for $a \leq x \leq b$. If E_M and E_T are the errors in the

Midpoint and Trapezoidal Rules, then $|E_M| \leq \frac{K(b-a)^3}{24n^2}$ and $|E_T| \leq \frac{K(b-a)^3}{12n^2}$. [Equations 3]



Simpson’s Rule: $\int_a^b f(x)dx \approx S_n := \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$, where n is even.

Error Bound for Simpson’s Rule: Suppose that $|f^{(4)}(x)| \leq K$, for $a \leq x \leq b$. If E_S is the error involved in using Simpson’s Rule, then $|E_S| \leq \frac{K(b-a)^5}{180n^4}$. [Equation 4]

Problem #6 Use the **Midpoint Rule** and **Simpson’s Rule** to approximate $\int_0^\pi x \cos x dx$, with $n = 4$. (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

$$\Delta x = \frac{b-a}{n} = \frac{\pi}{4}.$$

$$\begin{aligned} M_4 &= \frac{\pi}{4} \left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right) \right] \\ &= \frac{\pi}{4} \left(\frac{\pi}{8} \cos \frac{\pi}{8} + \frac{3\pi}{8} \cos \frac{3\pi}{8} + \frac{5\pi}{8} \cos \frac{5\pi}{8} + \frac{7\pi}{8} \cos \frac{7\pi}{8} \right) \approx -1.945744 \end{aligned}$$

$$\begin{aligned} S_4 &= \frac{\pi}{4 \cdot 3} \left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{2\pi}{4}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right] \\ &= \frac{\pi}{12} \left(0 + \pi \cos \frac{\pi}{4} + \pi \cos \frac{\pi}{2} + 3\pi \cos \frac{3\pi}{4} + \pi \cos \pi \right) \approx -1.985611 \end{aligned}$$

Exact area under the curve: ??

$$\begin{aligned} &\int_0^\pi x \cos x dx \\ &= [x \sin x]_0^\pi - \int_0^\pi \sin x dx = [x \sin x + \cos x]_0^\pi \\ &= (0 + (-1)) - (0 + 1) = -2. \end{aligned}$$

Errors: $E_M = \text{exact} - M_4 \approx (-2) - (-1.945744) = -0.054256,$

$E_S = \text{exact} - S_4 \approx (-2) - (-1.985611) = -0.014389.$

Problem #12 Use of the Trapezoidal Rule, the Midpoint Rule, and Simpson’s rule to approximate $\int_1^3 e^{\frac{1}{x}} dx,$ with $n = 8.$
(Round your answers to six decimal places.)

$\Delta x = \frac{3-1}{8} = \frac{1}{4}.$

$T_8 = \frac{1}{4 \cdot 2} \Big[f(1) + 2f\Big(\frac{5}{4}\Big) + 2f\Big(\frac{3}{2}\Big) + 2f\Big(\frac{7}{4}\Big) + 2f(2) + 2f\Big(\frac{9}{4}\Big) + 2f\Big(\frac{5}{2}\Big) + 2f\Big(\frac{11}{4}\Big) + f(3) \Big]$

$= \frac{1}{8} \Big[e + 2e^{\frac{4}{5}} + 2e^{\frac{2}{3}} + 2e^{\frac{4}{7}} + 2e^{\frac{1}{2}} + 2e^{\frac{4}{9}} + 2e^{\frac{2}{5}} + 2e^{\frac{4}{11}} + e^{\frac{1}{3}} \Big] \approx 3.534934.$

$M_8 = \frac{1}{4} \Big[f\Big(\frac{9}{8}\Big) + f\Big(\frac{11}{8}\Big) + f\Big(\frac{13}{8}\Big) + f\Big(\frac{15}{8}\Big) + f\Big(\frac{17}{8}\Big) + f\Big(\frac{19}{8}\Big) + f\Big(\frac{21}{8}\Big) + f\Big(\frac{23}{8}\Big) \Big]$

$= \frac{1}{4} \Big[e^{\frac{8}{9}} + e^{\frac{8}{11}} + e^{\frac{8}{13}} + e^{\frac{8}{15}} + e^{\frac{8}{17}} + e^{\frac{8}{19}} + e^{\frac{8}{21}} + e^{\frac{8}{23}} \Big] \approx 3.515248.$

$S_8 = \frac{1}{4 \cdot 3} \Big[f(1) + 4f\Big(\frac{5}{4}\Big) + 2f\Big(\frac{3}{2}\Big) + 4f\Big(\frac{7}{4}\Big) + 2f(2) + 4f\Big(\frac{9}{4}\Big) + 2f\Big(\frac{5}{2}\Big) + 4f\Big(\frac{11}{4}\Big) + f(3) \Big]$

$= \frac{1}{12} \Big[e + 4e^{\frac{4}{5}} + 2e^{\frac{2}{3}} + 4e^{\frac{4}{7}} + 2e^{\frac{1}{2}} + 4e^{\frac{4}{9}} + 2e^{\frac{2}{5}} + 4e^{\frac{4}{11}} + e^{\frac{1}{3}} \Big] \approx 3.522375.$

Problem #21 a) Find the approximations $T_{10}, M_{10},$ and S_{10} for $\int_0^{\pi} \sin x dx$ and the corresponding errors $E_T, E_M,$ and $E_S.$

$T_{10} = \frac{\pi}{10 \cdot 2} \Big[f(0) + 2f\Big(\frac{\pi}{10}\Big) + 2f\Big(\frac{2\pi}{10}\Big) + 2f\Big(\frac{2\pi}{10}\Big) + \dots + 2f\Big(\frac{9\pi}{10}\Big) + f(\pi) \Big]$

$= \frac{\pi}{20} \Big[\sin 0 + 2 \sin \frac{\pi}{10} + 2 \sin \frac{2\pi}{10} + 2 \sin \frac{2\pi}{10} + \dots + 2 \sin \frac{9\pi}{10} + \sin \pi \Big] \approx 1.983524.$

$M_{10} = \frac{\pi}{10} \Big[f\Big(\frac{\pi}{20}\Big) + f\Big(\frac{3\pi}{20}\Big) + f\Big(\frac{5\pi}{20}\Big) + \dots + f\Big(\frac{19\pi}{20}\Big) \Big]$

$= \frac{\pi}{10} \Big[\sin \frac{\pi}{20} + \sin \frac{3\pi}{20} + \sin \frac{5\pi}{20} + \dots + \sin \frac{19\pi}{20} \Big] \approx 2.008248.$

$S_{10} = \frac{\pi}{10 \cdot 3} \Big[f(0) + 4f\Big(\frac{\pi}{10}\Big) + 2f\Big(\frac{2\pi}{10}\Big) + 4f\Big(\frac{3\pi}{10}\Big) + \dots + 4f\Big(\frac{9\pi}{10}\Big) + f(\pi) \Big]$

$= \frac{\pi}{30} \Big[\sin 0 + 4 \sin \frac{\pi}{10} + 2 \sin \frac{2\pi}{10} + 4 \sin \frac{3\pi}{10} + \dots + 4 \sin \frac{9\pi}{10} + \sin \pi \Big] \approx 2.000110.$

Exact area under the curve: $\int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1 - (-1) = 2.$

So, $E_T = \text{ exact } - T_{10} \approx 2 - 1.983524 = 0.016476,$

$E_M \approx 2 - 2.008248 = -0.008248,$ and $E_S \approx 2 - 2.000110 = -0.000110.$

b) Compare the actual errors in part a) with the error estimates given by equations 3 and 4 in the review.

Since $f(x) = \sin x,$ we have $|f^{(n)}(x)| \leq 1,$ so we can use $K = 1$ for all our (upper) error estimates.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{1(\pi-0)^3}{12(10)^2} = \frac{\pi^3}{1200} \approx 0.025839$$

$$|E_M| \leq \frac{|E_T|}{2} = \frac{\pi^3}{2400} \approx 0.012919.$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{1(\pi-0)^5}{180(10)^4} = \frac{\pi^5}{1,800,000} \approx 0.000170.$$

The actual error is about 64% of the error estimate in all three cases.

c) How large do we have to choose n so that the approximations $T_n,$ $M_n,$ and S_n to the integral in part a) are accurate to within 0.00001?

$$|E_T| \leq 0.00001 \text{ gives us: } \frac{\pi^3}{12n^2} \leq \frac{1}{10^5} \text{ or } n^2 \geq \frac{10^5\pi^3}{12} \Rightarrow n \geq 508.3.$$

So we can take $n = 509$ for $T_n.$

$$|E_M| \leq 0.00001 \text{ gives us: } \frac{\pi^3}{24n^2} \leq \frac{1}{10^5} \text{ or } n^2 \geq \frac{10^5\pi^3}{24} \Rightarrow n \geq 359.4.$$

So we can take $n = 360$ for $M_n.$

$$|E_S| \leq 0.00001 \text{ gives us: } \frac{\pi^5}{180n^4} \leq \frac{1}{10^5} \text{ or } n^4 \geq \frac{10^5\pi^5}{180} \Rightarrow n \geq 20.3.$$

So we can take $n = 22$ for S_n (since n must be even).

Problem #28 Given $\int_1^4 \frac{1}{\sqrt{x}} dx,$ find the approximations $T_n,$ $M_n,$ and S_n for $n = 6$ and $n = 12.$ Then compute the corresponding errors $E_T,$ $E_M,$ and $E_S.$ (Round your answers to six decimal places.) What observations can you make? In particular, what happens to the errors when n is doubled?

Exact: $\int_1^4 \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^4 = 4 - 2 = 2, \qquad \Delta x = \frac{4-1}{n} = \frac{3}{n}.$

For $n = 6 :$

$$\begin{aligned}
T_6 &= \frac{3}{6.2} \{f(1) + 2[f(\frac{3}{2}) + f(\frac{4}{2}) + f(\frac{5}{2}) + f(\frac{6}{2}) + f(\frac{7}{2})] + f(4)\} \\
&= \frac{1}{4} \left\{1 + 2\left[\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{5}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{2}{7}}\right] + \frac{1}{2}\right\} \approx 2.008966.
\end{aligned}$$

$$\begin{aligned}
M_6 &= \frac{3}{6} [f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4}) + f(\frac{13}{4}) + f(\frac{15}{4})] \\
&= \frac{1}{2} \left[\sqrt{\frac{4}{5}} + \sqrt{\frac{4}{7}} + \sqrt{\frac{4}{9}} + \sqrt{\frac{4}{11}} + \sqrt{\frac{4}{13}} + \sqrt{\frac{4}{15}}\right] \approx 1.995572.
\end{aligned}$$

$$\begin{aligned}
S_6 &= \frac{3}{6.3} [f(1) + 4f(\frac{3}{2}) + 2f(\frac{4}{2}) + 4f(\frac{5}{2}) + 2f(\frac{6}{2}) + 4f(\frac{7}{2}) + f(4)] \\
&= \frac{1}{6} \left[1 + 4\sqrt{\frac{2}{3}} + 2\sqrt{\frac{1}{2}} + 4\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{3}} + 4\sqrt{\frac{2}{7}} + \frac{1}{2}\right] \approx 2.000469.
\end{aligned}$$

$$E_T = \text{ exact } - T_6 \approx 2 - 2.008966 = -0.008966,$$

$$E_M \approx 2 - 1.995572 = 0.004428,$$

$$E_S \approx 2 - 2.000469 = -0.000469.$$

For $n = 12$:

$$\begin{aligned}
T_{12} &= \frac{3}{12.2} \{f(1) + 2[f(\frac{5}{4}) + f(\frac{6}{4}) + f(\frac{7}{4}) + \dots + f(\frac{15}{4})] + f(4)\} \\
&= \frac{1}{8} \left\{1 + 2\left[\sqrt{\frac{4}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{4}{7}} + \dots + \sqrt{\frac{4}{15}}\right] + \frac{1}{2}\right\} \approx 2.002269,
\end{aligned}$$

$$\begin{aligned}
M_{12} &= \frac{3}{12} [f(\frac{9}{8}) + f(\frac{11}{8}) + f(\frac{13}{8}) + \dots + f(\frac{31}{8})] \\
&= \frac{1}{4} \left[\sqrt{\frac{8}{9}} + \sqrt{\frac{8}{11}} + \sqrt{\frac{8}{13}} + \dots + \sqrt{\frac{8}{31}}\right] \approx 1.998869,
\end{aligned}$$

$$\begin{aligned}
S_{12} &= \frac{3}{12.3} [f(1) + 4f(\frac{5}{4}) + 2f(\frac{6}{4}) + 4f(\frac{7}{4}) + 2f(\frac{8}{4}) + \dots + 4f(\frac{15}{4}) + f(4)] \\
&= \frac{1}{12} \left[1 + 4\sqrt{\frac{4}{5}} + 2\sqrt{\frac{2}{3}} + 4\sqrt{\frac{4}{7}} + 2\sqrt{\frac{1}{2}} + \dots + 4\sqrt{\frac{4}{15}} + \frac{1}{2}\right] \approx 2.000036.
\end{aligned}$$

$$E_T = \text{ exact } - T_{12} \approx 2 - 2.002269 = -0.002269,$$

$$E_M \approx 2 - 1.998869 = 0.001131,$$

$$E_S \approx 2 - 2.000036 = -0.000036.$$

Putting these results into tables:

n	T_n	M_n	S_n
6	2.008966	1.995572	2.000469
12	2.002269	1.998869	2.000036

n	E_T	E_M	E_S
6	-0.008966	0.004428	-0.000469
12	-0.002269	0.001131	-0.000036

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson’s approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.