

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.4 - Integration by Partial Fractions

Improper Rational Function: Say we need to integrate a function of the form $f(x) = \frac{P(x)}{Q(x)}$, for polynomials $P(x)$ and $Q(x)$. If $\deg(P) \geq \deg(Q)$, use polynomial division to find: $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where $\deg(R) < \deg(Q)$.

Since all polynomials, and hence $S(x)$ is easily integrated, the problem reduces to integrating $\frac{R(x)}{Q(x)}$. How do we do this?

Case I: If the denominator is a product of distinct linear factors: $Q(x) = (ax + b)(cx + d)\dots(ex + f)$, where no factor is repeated, and no factor is a constant multiple of another:

1. Set: $\frac{R(x)}{Q(x)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \dots + \frac{C}{ex+f}$
2. Multiply the equation through by $Q(x)$
3. Collect like terms on the right-hand side of the equation, and generate a system of equations by equating the coefficients of equal powers of x .
4. Solve the system of equations, and replace A, B, C in the above equation with their values.
5. Integrate this simpler expression.

Example: $\frac{3x+1}{(7x+2)(4x+13)} = \frac{A}{7x+2} + \frac{B}{4x+13}$

$$\begin{aligned} 3x + 1 &= A(4x + 13) + B(7x + 2) && \text{[multiplied by } Q(x)\text{]} \\ &= 4Ax + 13A + 7Bx + 2B = (4A + 7B)x + (13A + 2B) && \text{[sorted by powers of } x\text{]} \\ \Rightarrow \quad 4A + 7B &= 3 && \text{and} \quad 13A + 2B = 1. && \text{[list of equations by power of } x\text{]} \end{aligned}$$

The last two steps above for this example are elementary.

Case II: If some of the linear factors in the above example are repeated:

For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{x^3(7x+2)^2(4x+13)}$.

♦ Set: $\frac{3x+1}{x^3(7x+2)^2(4x+13)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{7x+2} + \frac{E}{(7x+2)^2} + \frac{F}{4x+13}$

♦ Continue with the procedure for Case I, from step 2.

Case III: The denominator contains irreducible quadratic factors, none of which are

repeated (This is the case when the discriminant of the quadratic ($b^2 - 4ac$) is less than zero.). For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{(x^2+2)(4x+13)}$.

♦ Set: $\frac{3x+1}{(x^2+2)(4x+13)} = \frac{Ax+B}{x^2+2} + \frac{C}{4x+13}$. [quadratic terms get linear numerators in x]

♦ Continue with the procedure for Case I, from step 2.

Case IV: The denominator contains a repeated irreducible quadratic factor

For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{(x^2+2)^2(4x+13)}$.

♦ Set: $\frac{3x+1}{(x^2+2)^2(4x+13)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{E}{4x+13}$. [similarly to Case II]

♦ Continue with the procedure for Case I, from step 2.

When integrating the resulting terms from Case III/IV, you will find the need to integrate terms in the form: $\frac{1}{x^2+a^2}$. So, a useful integration formula to recall is: $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$.

Rationalizing Substitutions:

When you have a function that contains a radical ($\sqrt[n]{g(x)}$), you can often transform the function into a rational function for which you can apply the above cases. You can test out if this is possible by substituting $u = \sqrt[n]{g(x)}$, and seeing if this results in a rational function.

Problem #8 Evaluate $\int \frac{3t-2}{t+1} dt$.

We hope for the following equality: $\frac{3t-2}{t+1} = \frac{A}{1} + \frac{B}{t+1}$, for some A, B .

If so, then: $3t - 2 = A(t + 1) + B$

$$3t - 2 = At + (A + B).$$

Comparing sides of the equation for powers of t we find:

$$A = 3, \quad A + B = -2, \quad \text{and so } B = -5.$$

Finally: $\int \frac{3t-2}{t+1} dt = \int \left(3 - \frac{5}{t+1}\right) dt$

$$= 3t - 5 \ln|t + 1| + C.$$

Problem #32 Evaluate $\int_0^1 \frac{x}{x^2+4x+13} dx$.

Your first impulse might be to complete the square in the denominator:

$$x^2 + 4x + 13 = (x + 2)^2 + 9. \text{ And we will use this!}$$

However, we need one other tool. Since we know that $(\ln y)' = \frac{y'}{y}$, one method for integrating is to see if you can make the numerator look like the derivative of the denominator. Because then we have $\int \frac{y'}{y} = \ln y$. In this case, the derivative of the denominator is $2x + 4$. To make the numerator look like this, we must multiply it by 2, and add 4. However, we must then do the inverse operations so we don't change our answer. Observe:

$$\int_0^1 \frac{x}{x^2+4x+13} dx = \int_0^1 \frac{\frac{1}{2}[(2x+4)-4]}{x^2+4x+13} dx = \frac{1}{2} \int_0^1 \frac{2x+4}{x^2+4x+13} dx - \int_0^1 \frac{2}{x^2+4x+13} dx$$

We are now in position to complete the first of the two resulting integrals.

Let $y = x^2 + 4x + 13$, so $dy = (2x + 4)dx$.

Now we have: $\frac{1}{2} \int_0^1 \frac{2x+4}{x^2+4x+13} dx = \frac{1}{2} \int_{y=13}^{y=18} \frac{dy}{y} = \frac{1}{2} [\ln y]_{13}^{18} = \frac{1}{2} \ln \frac{18}{13}$.

With the second integral, we will complete the square

$$-\int_0^1 \frac{2}{x^2+4x+13} dx = -2 \int_0^1 \frac{1}{9+(x+2)^2} dx.$$

Now from this, hopefully you can see the connection to $\tan^{-1}f(x) = \int \frac{f'}{1+f^2} dx$.

However, to get the 9 transformed into a one, we're going to have to pull a 9 out of the squared term in the denominator.

$$\text{Observe that } -2 \int_0^1 \frac{1}{9+(x+2)^2} dx = -2 \int_0^1 \left(\frac{\frac{1}{9}}{\frac{1}{9} + \frac{(x+2)^2}{9}} \right) dx = -\frac{2}{9} \int_0^1 \frac{1}{1+\frac{(x+2)^2}{9}} = -\frac{2}{3} \int_0^1 \frac{\frac{1}{3}}{1+\left(\frac{x+2}{3}\right)^2}$$

So using our above identity, we consider $f = \frac{x+2}{3}$ and we have:

$$= -\frac{2}{3} \int \frac{f'}{1+f^2} dx = -\frac{2}{3} \int_0^1 \frac{\frac{1}{3}}{1+\left(\frac{x+2}{3}\right)^2} = -\frac{2}{3} \tan^{-1} \frac{x+2}{3} \Big|_0^1 = -\frac{2}{3} \tan^{-1} 1 + \frac{2}{3} \tan^{-1} \frac{2}{3}$$

$$= -\frac{2}{3} \cdot \frac{\pi}{4} + \frac{2}{3} \tan^{-1} \frac{2}{3}.$$

Recall that $\frac{\pi}{4}$ is the angle you need to feed into $\tan x = \frac{\sin x}{\cos x}$ in order to get $\tan \frac{\pi}{4} = \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = 1$.

So $\tan^{-1} 1 = \frac{\pi}{4}$.

Finally, putting the first and second in together we have:

$$\int_0^1 \frac{x}{x^2+4x+13} dx = \left(\frac{1}{2} \ln \frac{18}{13} \right) + \left(\frac{2}{3} \tan^{-1} \frac{2}{3} - \frac{\pi}{6} \right).$$

Problem #43 Make a substitution to express $\int \frac{x^3}{\sqrt[3]{x^2+1}} dx$ as a rational function and then evaluate the integral.

Let $u = \sqrt[3]{x^2+1}$.

Your first impulse might then be to take the derivative of both sides, but you might observe in advance that this will be messy. To get rid of the radical sign, it is helpful to cube both sides first. Then, $x^2 = u^3 - 1$, and $2xdx = 3u^2 du$.

$$\text{So: } \int \frac{x^3}{\sqrt[3]{x^2+1}} dx = \int \frac{(u^3-1)^{\frac{3}{2}} u^2}{u} du = \frac{3}{2} \int (u^4 - u) du$$

$$= \frac{3}{10} u^5 - \frac{3}{4} u^2 + C = \frac{3}{10} (x^2+1)^{\frac{5}{3}} - \frac{3}{4} (x^2+1)^{\frac{2}{3}} + C.$$