

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.3 - Trigonometric Substitutions

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Used to make inverse substitutions.

Problem #2 Evaluate $\int \frac{x^3}{\sqrt{x^2+4}} dx$ using the trigonometric substitution $x = 2 \tan \theta$. Sketch and label the associated right triangle.

Let $x = 2 \tan \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then, $dx = 2 \sec^2 \theta d\theta$.

So, $\sqrt{x^2 + 4} = \sqrt{4 \tan^2 \theta + 4} = \sqrt{4(\tan^2 \theta + 1)} = \sqrt{4 \sec^2 \theta} = 2|\sec \theta| = 2 \sec \theta$ on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\int \frac{x^3}{\sqrt{x^2+4}} dx = \int \frac{8 \tan^3 \theta}{2 \sec \theta} (2 \sec^2 \theta d\theta) = 8 \int \tan^3 \theta \sec \theta d\theta \quad [\text{this should look familiar!}]$$

$$= 8 \int \tan^2 \theta \sec \theta \tan \theta d\theta = 8 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta$$

Let $u = \sec \theta$ and $du = \sec \theta \tan \theta$

$$\text{So, } \int \frac{x^3}{\sqrt{x^2+4}} dx = 8 \int (u^2 - 1) du$$

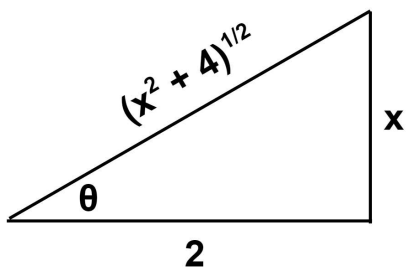
$$= 8 \left(\frac{1}{3} u^3 - u \right) + C \quad [\text{am I done?}]$$

$$= 8 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) + C. \quad [\text{how about now?}]$$

Observe the from above that $\sec \theta = \frac{\sqrt{x^2+4}}{2}$.

$$\text{Therefore, } \int \frac{x^3}{\sqrt{x^2+4}} dx = 8 \left(\frac{1}{3} \left(\frac{\sqrt{x^2+4}}{2} \right)^3 - \frac{\sqrt{x^2+4}}{2} \right) + C$$

$$= 8 \left(\frac{(x^2+4)^{\frac{3}{2}}}{3 \cdot 8} - \frac{(x^2+4)^{\frac{1}{2}}}{2} \right) + C = \frac{1}{3} (x^2+4)^{\frac{3}{2}} - 4(x^2+4)^{\frac{1}{2}} + C.$$



Problem #30 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}} dt$.

Let $u = \sin t$, $du = \cos t dt$.

$$\text{Then, } \int_{t=0}^{t=\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}} dt = \int_{u=0}^{u=1} \frac{1}{\sqrt{1+u^2}} du.$$

Now let $u = \tan \theta$.

$$\text{Then we have: } \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sqrt{\sec^2 \theta} = \sec \theta.$$

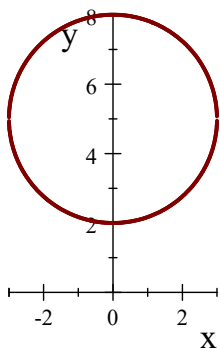
Also, $du = \sec^2 \theta d\theta$. So then:

$$\int_{u=0}^{u=1} \frac{1}{\sqrt{1+u^2}} du = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{1}{\sec \theta} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{4}} \sec \theta d\theta$$

$$= [\ln|\sec \theta + \tan \theta|]_0^{\frac{\pi}{4}} \quad (\text{by (1) in section 7.2})$$

$$= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1).$$

Problem #41 A torus (donut) is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x -axis. Find the volume enclosed by the torus.



Example: $R = 5$ and $r = 3$

We use cylindrical shells and assume that $R > r$.

$$x^2 = r^2 - (y - R)^2 \text{ implies } x = \pm \sqrt{r^2 - (y - R)^2}.$$

$$\text{So } g(y) = 2\sqrt{r^2 - (y - R)^2} \text{ and } V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y - R)^2} dy$$

$$= \int_{-r}^r 4\pi(u + R) \sqrt{r^2 - u^2} du \quad (\text{ where } u = y - R)$$

$$= 4\pi \int_{-r}^r u \sqrt{r^2 - u^2} du + 4\pi R \int_{-r}^r \sqrt{r^2 - u^2} du \quad (\text{ where } u = r \sin \theta, \quad du = r \cos \theta d\theta)$$

$$= 4\pi \left[-\frac{1}{3}(r^2 - u^2)^{\frac{3}{2}} \right]_{-r}^r + 4\pi R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2 \theta d\theta = -\frac{4\pi}{3}(0 - 0) + 4\pi R r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$V = 2\pi R r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2\pi R r^2 \left[\theta + \frac{1}{2 \sin} 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi^2 R r^2.$$

Another Method: Use washers instead of shells, so $V = 8\pi R \int_0^r \sqrt{r^2 - y^2} dy$, and evaluate integral using $y = r \sin \theta$.