

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.2 - Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use

the substitution $\cos^2 x = 1 - \sin^2 x$ (recall $\sin^2 x + \cos^2 x = 1$) to express the

remaining factors in terms of sine: $\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx.$$

Then, substitute $u = \sin x$.

b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use

$\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx.$$

Then substitute $u = \cos x$. [Note that if the powers of **both** sine and cosine are

odd, either a) or b) can be used.]

c) If the powers of both sine and cosine are even, use the half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

It is also sometimes useful to use the double-angle identities:

$$\sin 2x = 2 \sin x \cos x \text{ and } \cos 2x = \cos^2 x - \sin^2 x.$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

a) If the power of secant is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use

$\sec^2 x = 1 + \tan^2 x$ (recall $\sec^2 x - \tan^2 x = 1$) to express the remaining factors

in terms of $\tan x$: $\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx$

$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx.$$

Then substitute $u = \tan x$.

b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use

$\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx.$$

Then, substitute $u = \sec x$.

Evaluating: $\int \sin mx \cos nx \, dx,$ $\int \sin mx \sin nx \, dx,$ $\int \cos mx \cos nx \, dx :$

Use the following trigonometric substitutions:

- a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)],$
- b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)],$
- c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)].$

Other Useful Trigonometric Integrals:

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Problem #2 Evaluate $\int \sin^3\theta \cos^4\theta \, d\theta.$

$$\begin{aligned} &= \int \sin^2\theta \cos^4\theta \sin\theta \, d\theta \\ &= \int (1 - \cos^2\theta) \cos^4\theta \sin\theta \, d\theta \\ u &= \cos\theta, \qquad du = -\sin\theta \, d\theta \\ &= \int (1 - u^2) u^4 (-du) = \int (u^6 - u^4) du \\ &= \frac{1}{7} u^7 - \frac{1}{5} u^5 + C = \frac{1}{7} \cos^7\theta - \frac{1}{5} \cos^5\theta + C. \end{aligned}$$

Problem #46 Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1 - \cos 4\theta} \, d\theta.$

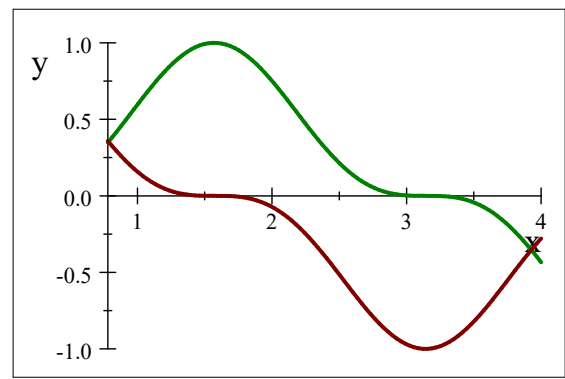
This looks difficult, and it would be better if the thing under the radical with the square of something, so we could eliminate the radical. Recall with trigonometric identities, we have ways of changing squared things into non-squared things, and vice versa. Recall the half-angle formula: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

So: $\cos 4\theta = 1 - 2 \sin^2(2\theta),$ giving us:

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \sqrt{1 - (1 - 2 \sin^2(2\theta))} \, d\theta = \int_0^{\frac{\pi}{4}} \sqrt{2 \sin^2(2\theta)} \, d\theta = \sqrt{2} \int_0^{\frac{\pi}{4}} \sqrt{\sin^2(2\theta)} \, d\theta \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} |\sin(2\theta)| \, d\theta \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} \sin 2\theta \, d\theta \quad [\text{since } \sin 2\theta \geq 0 \text{ for } 0 \leq \theta \leq \frac{\pi}{4}] \\ &= \sqrt{2} \left[-\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{4}} = \sqrt{2} \left(0 - \frac{1}{2} \right) = -\frac{\sqrt{2}}{2}. \end{aligned}$$

Problem # 58 Find the area of the region bounded by the curves:

$$y = \sin^3 x, \quad y = \cos^3 x, \quad \frac{\pi}{4} \leq x \leq \frac{5\pi}{4}.$$



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin^3 x - \cos^3 x) dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin^3 x dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos^3 x dx$$

$$= \int_{x=\frac{\pi}{4}}^{x=\frac{5\pi}{4}} (1 - \cos^2 x) \sin x dx - \int_{x=\frac{\pi}{4}}^{x=\frac{5\pi}{4}} (1 - \sin^2 x) \cos x dx$$

Let $u = \cos x$, $du = -\sin x dx$; and $v = \sin x$, $dv = \cos x dx$.

$$= \int_{u=\frac{\sqrt{2}}{2}}^{u=-\frac{\sqrt{2}}{2}} (1 - u^2)(-du) - \int_{v=\frac{\sqrt{2}}{2}}^{v=-\frac{\sqrt{2}}{2}} (1 - v^2) dv$$

$$= 2 \int_0^{\frac{\sqrt{2}}{2}} (1 - u^2) du + 2 \int_0^{\frac{\sqrt{2}}{2}} (1 - v^2) dv = 4 \left[u - \frac{1}{3} u^3 \right]_0^{\frac{\sqrt{2}}{2}}$$

$$= 4 \left[\left(\frac{\sqrt{2}}{2} - \frac{1}{3} \cdot \frac{2\sqrt{2}}{8} \right) - 0 \right] = 2\sqrt{2} - \frac{1}{3} \sqrt{2} = \frac{5}{3} \sqrt{2}.$$