

Calculus II

Textbook: *Calculus Early Transcendentals*, Stewart, (8th edition).

7.1 - Integration by Parts

The most important integrals you should have run across so far...

$$\begin{array}{ll} \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1) & \int \frac{1}{x} dx = \ln|x| + C \\ \int e^x dx = e^x + C & \int a^x dx = \frac{a^x}{\ln a} + C \\ \int \sin x \, dx = -\cos x + C & \int \cos x \, dx = \sin x + C \\ \int \sec^2 x \, dx = \tan x + C & \int \csc^2 x \, dx = -\cot x + C \\ \int \sec x \tan x \, dx = \sec x + C & \int \csc x \cot x \, dx = -\csc x + C \\ \int \sinh x \, dx = \cosh x + C & \int \cosh x \, dx = \sinh x + C \\ \int \tan x \, dx = \ln|\sec x| + C & \int \cot x \, dx = \ln|\sin x| + C \\ \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C & \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0 \end{array}$$

Integration by Parts (w/ indefinite integral):

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad (1)$$

OR

$$\int u \, dv = uv - \int v \, du \quad (2)$$

Integration by Parts (w/ definite integral):

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Problem #6 Evaluate $\int (x-1) \sin(\pi x) dx$.

Let $u = x-1$, and $dv = \sin(\pi x) dx$.

So, $du = dx$, and $v = -\frac{1}{\pi} \cos(\pi x)$.

Then by Equation 2 above, $\int (x-1) \sin(\pi x) dx = -\frac{1}{\pi} (x-1) \cos \pi x - \int \left(-\frac{1}{\pi} \cos(\pi x) dx\right)$

$$= -\frac{1}{\pi} (x-1) \cos(\pi x) + \frac{1}{\pi} \int \cos(\pi x) dx$$

$$= -\frac{1}{\pi} (x-1) \cos(\pi x) + \frac{1}{\pi^2} \sin(\pi x) + C.$$

Problem #36 Evaluate $\int_0^t e^s \sin(t-s) ds$.

Let $u = \sin(t-s)$, and $dv = e^s ds$.

So, $du = -\cos(t-s)ds$, and $v = e^s$.

$$\begin{aligned} \text{Then } I &= \int_0^t e^s \sin(t-s) ds = [e^s \sin(t-s)]_0^t + \int_0^t e^s \cos(t-s) ds \\ &= e^t \sin 0 - e^0 \sin t + I_1 = -\sin t + I_1. \end{aligned}$$

For I_1 , let $U = \cos(t-s)$ and $dV = e^s ds$.

So, $dU = \sin(t-s)ds$, and $V = e^s$.

$$\begin{aligned} \text{So } I_1 &= [e^s \cos(t-s)]_0^t - \int_0^t e^s \sin(t-s) ds = e^t \cos 0 - e^0 \cos t - I \\ &= e^t - \cos t - I. \end{aligned}$$

Thus, $I = -\sin t + I_1 = -\sin t + e^t - \cos t - I$.

$$\text{Or, } 2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}(e^t - \cos t - \sin t).$$

Problem #40 First make a substitution and then use integration by parts to evaluate $\int_0^\pi e^{\cos t} \sin 2t dt$.

Let $x = \cos t$, so that $dx = -\sin t dt$.

And $\sin 2t = 2 \sin t \cos t$.

$$\begin{aligned} \text{Thus, } \int_0^\pi e^{\cos t} \sin 2t dt &= \int_{t=0}^{t=\pi} e^x (2 \sin t \cos t) \left(\frac{-dx}{-\sin t} \right) \\ &= -2 \int_1^{-1} e^x \cos t dx \\ &= 2 \int_{-1}^1 x e^x dx. \end{aligned}$$

Now use IBPs with $u = x$, $dv = e^x dx$, which gives us $du = dx$, $v = e^x$, so we get:

$$2 \int_{-1}^1 x e^x dx = 2 \left([x e^x]_{-1}^1 - \int_{-1}^1 e^x dx \right)$$

$$= 2\left((e^1 - (-e^{-1})) - [e^x]_{-1}^1\right)$$

$$= 2\left((e + \frac{1}{e}) - [e - \frac{1}{e}]\right) = 2\left(\frac{2}{e}\right) = \frac{4}{e}.$$

Problem #52 Use integration by parts to prove the following reduction formula:

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

Let $u = x^n$ and $dv = e^x dx$.

This implies $du = nx^{n-1} dx$, and $v = e^x$.

By equation 2 we have, $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$