

# Exercises

## §8.2: Orthogonal Projection and Data

**Python:** There is an optional starter script that supports Python-based problem(s) on this sheet. Use it as scaffolding: read the comments, complete the lines marked **TODO**, and compare the output with the mathematics requested in the problem(s).

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**Problem 1** Let  $\mathbf{x} = (3, 1)^T$  and  $\mathbf{u} = (1, 1)^T / \sqrt{2}$ . Compute: (a) the coordinate  $z = \mathbf{u}^T \mathbf{x}$ , (b) the projection  $\hat{\mathbf{x}}$ , (c) the residual  $\mathbf{e}$ , (d) verify  $\mathbf{u}^T \mathbf{e} = 0$ , (e) verify the Pythagorean identity.

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**Problem 2** Project  $\mathbf{x} = (3, 4)^T$  onto the line spanned by  $\mathbf{v} = (2, 1)^T$ . Do not normalize  $\mathbf{v}$  first. Find the projection coefficient, projected vector, and residual. Verify that the residual is orthogonal to  $\mathbf{v}$ .

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**Problem 3** Let  $\mathbf{u} = (1, 1)^T / \sqrt{2}$ . Find  $P = \mathbf{u}\mathbf{u}^T$ . Verify directly that  $P^T = P$  and  $P^2 = P$ . What does  $I - P$  project onto?

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**Problem 4** Let  $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $\mathbf{x} = (2, -1, 5)^T$ . Compute: (a)  $\mathbf{z} = U^T \mathbf{x}$ , (b)  $\hat{\mathbf{x}} = UU^T \mathbf{x}$ , (c) the residual, (d) verify  $U^T \mathbf{e} = \mathbf{0}$ .

**Problem 5** Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$  and  $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ . Use the normal equations  $A^T A \mathbf{z} = A^T \mathbf{x}$  to find the projection of  $\mathbf{x}$  onto  $\text{Col}(A)$ . Verify that the residual is orthogonal to both columns of  $A$ .

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**Problem 6** Center the four observations  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 2)$ ,  $(3, 3)$ . Find the mean vector and all four centered observations. Verify that their average is  $(0, 0)$ .

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**Problem 7** For the centered data in Problem 6, project onto  $\mathbf{u} = (1, 1)^T / \sqrt{2}$ . Compute: (a) the four scalar coordinates, (b) the four reconstructed centered points, (c) the total squared reconstruction error, (d) the mean squared reconstruction distance. Then compare with projection onto  $\mathbf{u} = (1, 0)^T$ . Which of these two directions is better?

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**Problem 8** Use Python to reproduce Problem 7. Use `np`, `np`, and matrix multiplication. Confirm numerically that the mean squared reconstruction distance is 0.125 for the diagonal direction and 0.5 for the  $x_1$ -axis.