

3.1 Inner Products

How can we measure the length of a function, or decide whether two functions are perpendicular?



The dot product gives us the starting point: it turns algebra into geometry by revealing lengths and angles.

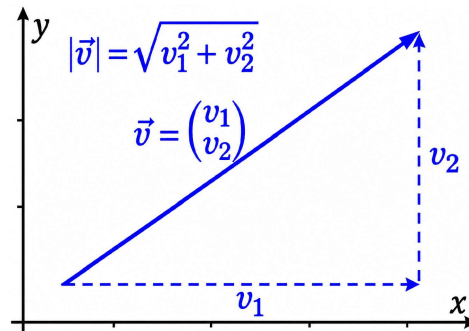
We extend that idea to "inner products," exploring how different choices of measurement give us useful ways to compare both vectors and functions.

Dot Product (or scalar product): $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i.$

In particular: $\vec{x} \cdot \vec{x} = x_1^2 + \dots + x_n^2.$

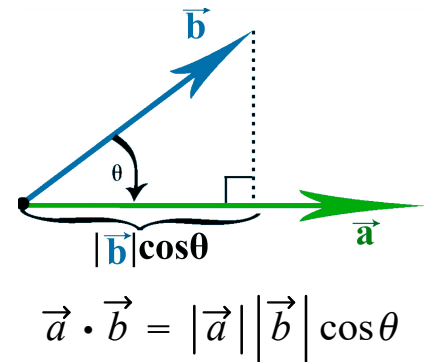
And so, the Pythagorean Thm comes from a dot product called

the *Euclidean Norm*: $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \dots + x_n^2}.$



Alternatively (and equivalently): If θ is angle between $\vec{a}, \vec{b}$ where $0 \leq \theta \leq \pi$, then $\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle = |\vec{a}| |\vec{b}| \cos \theta$. If $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

Observation: If θ is the angle between (nonzero) vectors $\vec{a}, \vec{b}$, then: $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}.$



! So the dot product can be used to determine the length of and angles between vectors.

More generally:

Definition: An *inner product* on the real vector space V is a pairing that takes two vectors $\vec{v}, \vec{w} \in V$;

produces a real number $\langle \vec{v}, \vec{w} \rangle \in \mathbb{R}$; and satisfies the following three axioms for all $\vec{u}, \vec{v}, \vec{w} \in V$, and scalars $c, d \in \mathbb{R}$.

♦ Bilinearity: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c\langle \vec{u}, \vec{w} \rangle + d\langle \vec{v}, \vec{w} \rangle$, and $\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c\langle \vec{u}, \vec{v} \rangle + d\langle \vec{u}, \vec{w} \rangle$.

♦ Symmetry: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.

♦ Positivity: $\langle \vec{v}, \vec{v} \rangle > 0$ whenever $\vec{v} \neq \vec{0}$. And $\langle \vec{0}, \vec{0} \rangle = 0$.

Check that the dot product satisfies these!

❗ There are inner products on $\mathbb{R}^n$ other than the dot product.

Definition: Given an inner product, the associated *norm* of $\vec{v} \in V$ is the positive square root of the inner product of the vector w/itself: $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

In Euclidean geometry, θ between two vectors can be formalized into the abstract concept of an inner product (defined below).

Inner products and norms lie at the heart of linear (and nonlinear) analysis, in both finite and infinite dimensional vector spaces.

Definition: A vector space equipped with an inner product and its associated norm is known as an **inner product space**.

Weighted Inner Product (a different inner product on $\mathbb{R}^n$)

$\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + 5v_2w_2$. One can verify this satisfies the axioms above.

The associated weighted norm: $|\vec{v}| = \sqrt{2v_1^2 + 5v_2^2}$ defines an alternative "non-Pythagorean" notion of length of vectors and distance between pts.

Example: Determine whether the following formulas for $\langle \vec{v}, \vec{w} \rangle$ define inner products on $\mathbb{R}^2$:

a) $\langle \vec{v}, \vec{w} \rangle = (v_1 + v_2)(w_1 + w_2)$.

Bilinearity? LHS is: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = (cu_1 + dv_1 + cu_2 + dv_2)(w_1 + w_2)$.

RHS is: $c\langle \vec{u}, \vec{w} \rangle + d\langle \vec{v}, \vec{w} \rangle = c(u_1 + u_2)(w_1 + w_2) + d(v_1 + v_2)(w_1 + w_2)$
 $= (cu_1 + dv_1 + cu_2 + dv_2)(w_1 + w_2)$. ✓

Symmetry: $\langle \vec{v}, \vec{w} \rangle = (v_1 + v_2)(w_1 + w_2) = (w_1 + w_2)(v_1 + v_2) = \langle \vec{w}, \vec{v} \rangle$. ✓

Positivity: $\langle \vec{v}, \vec{v} \rangle = (v_1 + v_2)(v_1 + v_2) = (v_1 + v_2)^2$. Observe that $\langle \vec{v}, \vec{v} \rangle$ is zero if $\vec{v} = 0$.

However, if $v_1 = -v_2 \neq 0$, then $\langle \vec{v}, \vec{v} \rangle = 0$, (e.g, $(1, -1)$). Therefore it does not pass positivity. ✗

Verdict: Not an inner product space. ✗

b) $\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + (v_1 - v_2)(w_1 - w_2)$

Bilinearity? LHS: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = 2w_1(cu_1 + dv_1) + ((cu_1 + dv_1) - (cu_2 + dv_2))(w_1 - w_2)$

RHS: $\langle c\vec{u}, \vec{w} \rangle + \langle d\vec{v}, \vec{w} \rangle = 2cu_1w_1 + c(u_1 - u_2)(w_1 - w_2) + 2dv_1w_1 + d(v_1 - v_2)(w_1 - w_2)$

$$= 2w_1(cu_1 + dv_1) + ((cu_1 + dv_1) - (cu_2 + dv_2))(w_1 - w_2) \quad \checkmark$$

$$\text{Symmetry: } \langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + (v_1 - v_2)(w_1 - w_2) = 2w_1v_1 + (w_1 - w_2)(v_1 - v_2) = \langle \vec{w}, \vec{v} \rangle \quad \checkmark$$

$$\text{Positivity: } \langle \vec{v}, \vec{v} \rangle = 2v_1^2 + (v_1 - v_2)^2. \text{ Observe that } \langle \vec{v}, \vec{v} \rangle \text{ is zero if } \vec{v} = \vec{0}.$$

Observe that both terms are nonnegative.

To be zero, we would need $v_1 = 0$ and $v_1 = v_2$, which require $v_2 = 0$. $\checkmark$

Verdict: It IS an inner product space. $\checkmark$

Inner Products on Function Spaces

Let $[a, b] \subseteq \mathbb{R}$ be a bounded, closed interval. The function space $C^0[a, b]$ consists of all continuous, scalar functions f defined on $[a, b]$.

Claim (proven below): an inner product of $C^0[a, b]$ can be defined by the integral of the product of two continuous functions f, g :

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

The associated norm is $|f| = \sqrt{\int_a^b f(x)^2 dx}$, and is known as the L^2 **norm** of the function f over the interval $[a, b]$.

The L^2 inner product and norm of these functions can be viewed as the infinite-dimensional function space versions of the dot product and Euclidean norm of the vectors in $\mathbb{R}^2$.



Example: If $[a, b] = [0, \frac{\pi}{2}]$, then L^2 inner product between $f(x) = \sin x$ and $g(x) = \cos x$ is equal to ...

$$\langle \sin x, \cos x \rangle = \int_0^{\frac{\pi}{2}} \sin x \cos x dx = \frac{1}{2} \sin^2 x \Big|_{x=0}^{\frac{\pi}{2}} = \frac{1}{2}. \quad (\text{u-sub w/u} = \sin x)$$

Norm of $\sin x$ is $|\sin x| = ??$

$$= \sqrt{\int_0^{\frac{\pi}{2}} (\sin x)^2 dx} = \dots = \sqrt{\frac{\pi}{4}}. \quad (\text{trig ident: } \sin^2 x = \frac{1 - \cos(2x)}{2})$$

 **Warning:** Observe: $|1| = \sqrt{\int_0^{\frac{\pi}{2}} 1^2 dx} = \sqrt{\frac{\pi}{2}} \neq 1$.

$$\text{On } [0, \pi]: |1| = \sqrt{\int_0^{\pi} 1^2 dx} = \sqrt{\pi}.$$

L^2 inner products & norms depend upon the interval of the function space!

Now we prove the above claim ($\langle f, g \rangle = \int_a^b f(x)g(x)dx$ is inner product):

First, is $\langle f, g \rangle = \int_a^b f(x)g(x)dx$ even **well defined**?

That is, does this formula assign a finite real number to every pair of functions in our space?

Notice that the product of two continuous functions is also continuous,
and thus the integral over a bounded interval is finite and real. ✓

Symmetric Requirement: Need: $\langle f, g \rangle = \langle g, f \rangle$.

$$\langle f, g \rangle = \int_a^b fg \, dx = \int_a^b gf \, dx = \langle g, f \rangle. \quad \checkmark$$

First Bilinearity Axiom: Need: $\langle cf + dg, h \rangle = c\langle f, h \rangle + d\langle g, h \rangle$. Follows from observing that

$$\int_a^b [cf(x) + dg(x)]h(x) \, dx = c \int_a^b f(x)h(x) \, dx + d \int_a^b g(x)h(x) \, dx. \quad \checkmark$$

Second Bilinearity Axiom: similarly proved. ✓

Positivity requires that $|f|^2 = \langle f, f \rangle = \int_a^b f(x)^2 \, dx \geq 0$.

But, observe that $f(x)^2 \geq 0$.

And since $f(x)^2$ is continuous, nonnegative, $|f|^2 = \int_a^b f(x)^2 \, dx = 0$ (norm is zero) **iff**

$$f(x) \equiv 0 \text{ (the zero function)}. \quad \checkmark$$



Warning: Be careful when extending L^2 inner product to other spaces of functions.

Nonzero discontinuous functions with zero " L^2 norm."

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{satisfies } |f|^2 = \int_{-1}^1 f(x)^2 \, dx = 0. \quad \text{So it does not satisfy positivity.}$$

L^2 is but one example of an inner product on function spaces.

Weighted inner product space on the space $C^0[a, b]$. For a continuous, positive, scalar function $w(x) > 0$.

$$\text{Weighted Inner Product and Norm: } \langle f, g \rangle = \int_a^b f(x)g(x)w(x) \, dx, \quad |f| = \sqrt{\int_a^b f(x)^2 w(x) \, dx}.$$

Weighted inner products used in statistics and data analysis.

Example: For $f(x) = \cos(2\pi x)$, and $g(x) = \sin(2\pi x)$ in $C^0[0, 1]$, find their L^2 inner product $\langle f, g \rangle$ and their L^2 norms $|f|, |g|$.

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 f(x)g(x)dx = \int_0^1 \cos(2\pi x) \sin(2\pi x)dx \\ &= \frac{1}{2} \int_0^1 \sin(4\pi x)dx = \frac{1}{8\pi} [-\cos(4\pi x)]_0^1 \qquad (\text{trig ident: } \sin(2\theta) = 2 \sin \theta \cos \theta) \\ &= \frac{1}{8\pi} (-\cos(4\pi) + \cos(0)) = \frac{1}{8\pi} (0) = 0. \end{aligned}$$

$$\begin{aligned} |f| &= \sqrt{\int_a^b f(x)^2 dx} = \sqrt{\int_0^1 \cos^2(2\pi x) dx} \\ &= \sqrt{\int_0^1 \frac{1+\cos(4\pi x)}{2} dx} \qquad (\text{trig ident: } \cos^2 \theta = \frac{1+\cos(2\theta)}{2}) \\ &= \sqrt{\left[\frac{1}{2}x + \frac{1}{8\pi} \sin(4\pi x) \right]_0^1} = \sqrt{\left(\frac{1}{2} + 0 \right) - (0 + 0)} = \frac{1}{\sqrt{2}}. \end{aligned}$$

$$\begin{aligned} |g| &= \sqrt{\int_a^b g(x)^2 dx} = \sqrt{\int_0^1 \sin^2(2\pi x) dx} \\ &= \sqrt{\int_0^1 \frac{1-\cos(4\pi x)}{2} dx} = \sqrt{\left[\frac{1}{2}x - \frac{1}{8\pi} \sin(4\pi x) \right]_0^1} = \sqrt{\left(\frac{1}{2} + 0 \right) - (0 + 0)} = \frac{1}{\sqrt{2}}. \qquad (\text{trig ident: } \sin^2 x = \frac{1-\cos(2x)}{2}) \end{aligned}$$

Example: For $f(x) = (x + 1)^2$, and $g(x) = \frac{1}{x+1}$ in $C^0[0, 1]$, find their L^2 inner product $\langle f, g \rangle$ and their L^2 norms $|f|, |g|$.

$$\begin{aligned} \langle f, g \rangle &= \int_a^b f(x)g(x)dx = \int_0^1 \frac{(x+1)^2}{x+1}dx \\ &= \int_0^1 (x + 1) dx = \left[\frac{1}{2}x^2 + x \right]_0^1 = \frac{3}{2}. \end{aligned}$$

$$\begin{aligned} |f| &= \sqrt{\int_a^b f(x)^2 dx} = \sqrt{\int_0^1 (x + 1)^4 dx} \\ &= \sqrt{\left[\frac{1}{5}(x + 1)^5 \right]_0^1} = \sqrt{\frac{2^5}{5} - \frac{1}{5}} = \sqrt{\frac{31}{5}}. \end{aligned}$$

$$\begin{aligned} |g| &= \sqrt{\int_a^b g(x)^2 dx} = \sqrt{\int_0^1 (x + 1)^{-2} dx} \\ &= \sqrt{\left[-(x + 1)^{-1} \right]_0^1} = \sqrt{-\frac{1}{2} + 1} = \frac{1}{\sqrt{2}}. \end{aligned}$$