

# Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

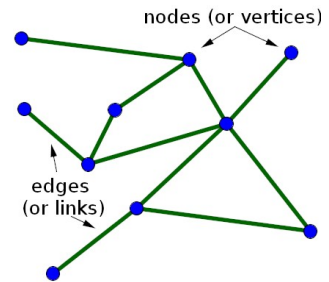
## 2.6 - Graphs and Digraphs

What do electrical circuits, road systems, and water pipelines have in common?

Each can be modeled as a network of pts joined by connections, a graph.

We'll turn these networks into matrices and discover that their kernels and cokernels reveal connected components and independent loops.

Foundation for analyzing how current, traffic, or water can flow through a network.



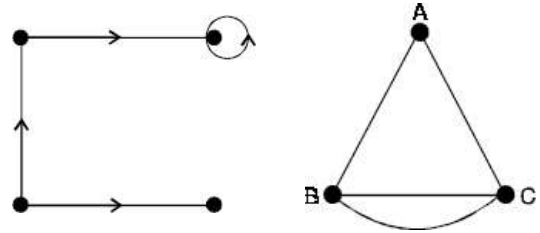
### Definitions:

*Graph*: finite # of points, called *vertices*, and finitely many lines or curves connecting them, called *edges*.

Each edge connects exactly two vertices, which are its *endpoints*.

To avoid technicalities, we'll always assume the graph is *simple*,  
i.e., that every edge connects two distinct vertices,  
so no edge forms a *loop* that connects a vertex to itself.

Moreover, we'll assume two distinct vertices are connected by,  
at most, one edge.



Non-Simple Graphs

Graphs encode **topology** (interconnectedness) of a system, but not its **geometry** (not the length of edges, angles, etc.).

Two graphs are the same if there's a **one-to-one correspondence** between their edges and their vertices,  
so that matched edges connect matched vertices.

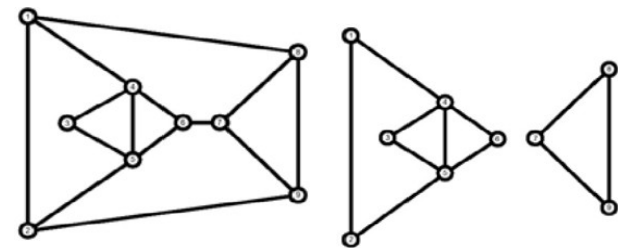
### Definitions:

A *path* in a graph is an ordered list of distinct edges  $e_1, \dots, e_k$  connecting vertices  $v_1, \dots, v_{k+1}$  so that edge  $e_i$  connects vertex  $v_i$  to  $v_{i+1}$ .

While an edge cannot be repeated in a path, a vertex may be.

A graph is *connected* if you can get from any vertex to any other vertex by a path.

A *circuit* (closed path) is a path that ends up where it began, i.e.,  $v_{k+1} = v_1$ .



Connected

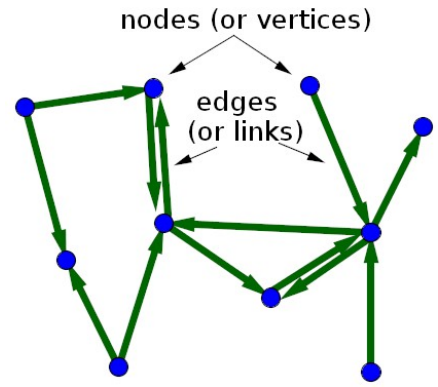
Disconnected

Two circuits that go around the edges in the same order, but w/two different starting points, are the **same circuit**.

A graph w/directed edges is known as a *directed graph*, or *digraph* for short.

The *Orientation* of an edge will be fixed by specifying the vertex the edge starts at, and the vertex it ends at.

The arrow on each edge sets a *direction*. When describing a circuit, we assign +1 to an edge traversed in the direction of its arrow, -1 to an edge traversed against its arrow, and 0 to an edge not used.

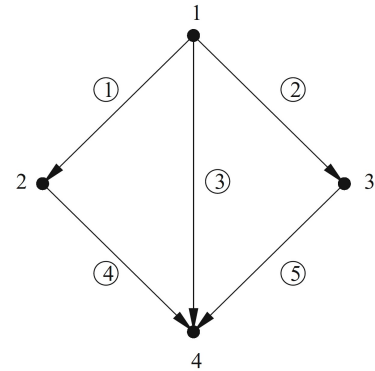


Directed Graph

Consider a digraph  $D$  consisting of  $n$  vertices connected by  $m$  edges.

The *incidence matrix* associated w/ $D$  is  $\mathbf{A}^{m \times n}$  whose rows are indexed by the edges and whose columns are indexed by the vertices.

It's a matrix consisting mostly of zeros, with 1s indicating where an edge begins, and -1s for the endpts.



Digraph  $D$

**Example:** The digraph above consists of five edges joined at four different vertices:

Its  $5 \times 4$  incidence matrix is:  $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ .

Thus the first row of  $\mathbf{A}$  tells us that the first edge starts at vertex 1 and ends at vertex 2, etc.

## Incidence Matrix Subspaces

Sure, we can describe a digraph w/an incidence matrix, but is this useful?

Yes, it encodes topological information!

## Kernel

**Proposition:** If  $\mathbf{A}$  is the incidence matrix for a *connected* digraph, then  $\ker \mathbf{A}$  is 1D, w/basis  $\vec{z} = (1, 1, \dots, 1)$ .

**Proof:** If edge  $k$  connects vertex  $i$  to vertex  $j$ , then the  $k^{\text{th}}$  eqn in  $\mathbf{A}\vec{z} = \vec{0}$  is  $z_i - z_j = 0$ , or, equivalently,  $z_i = z_j$ .

The same equality holds (by a simple induction) if the vertices  $i$  and  $j$  are connected by a **path**.

Therefore, if  $D$  is **connected**, then all the entries of  $\vec{z}$  are equal, and  $\ker \mathbf{A} = \text{span}\{(1, 1, \dots, 1)\}$ . ■

❗ In general,  $\dim \ker \mathbf{A}$  equals the number of connected components in a digraph  $D$ .

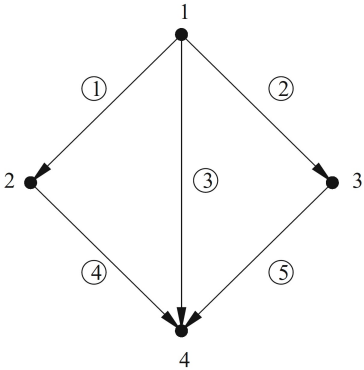
Applying FToLA (the fundamental theorem of linear algebra), we find:

**Corollary:** If  $\mathbf{A}$  is the incidence matrix for a connected digraph w/ $n$  vertices, then  $\text{rank } \mathbf{A} = n - 1$ .

### Cokernel

Looking at our previous example, the cokernel would come from calculating the kernel of the transposed matrix:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$



Digraph  $D$

Solving this we find the kernel spanned by  $\vec{y}_1 = (1, 0, -1, 1, 0)$  and  $\vec{y}_2 = (0, 1, -1, 0, 1)$ .

**Result:** Observe that each vector represents a *circuit* in the digraph. Recall entries are indexed by edges.  
Nonzero entry indicates the direction to traverse the corresponding edge.

For example,  $\vec{y}_1$  corresponds to the circuit that starts out along edge 1, then goes along edge 4, and finishes by going along edge 3 in the reverse direction.

If  $y_1$  and  $y_2$  are linearly independent vectors, we say the two circuits are “*independent*”.

**Theorem:** Each circuit in a digraph  $D$  is represented by a vector in the cokernel of its incidence matrix  $\mathbf{A}$ , whose entries are +1 if the edge is traversed in the set direction, -1 if in the opposite direction, and 0 if the edge is not in the circuit. The dimension of the cokernel of  $\mathbf{A}$  equals the number of independent circuits in  $D$ .

Observe that  $\vec{y}_1 - \vec{y}_2 = (1, 0, -1, 1, 0) - (0, 1, -1, 0, 1) = (1, -1, 0, 1, -1)$ ,  
another circuit around the edges as a linear combination of the two independent circuits.

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**Euler’s Formula for Graphs:** Suppose  $D$  is a connected digraph with  $m$  edges and  $n$  vertices and  $\mathbf{A}^{m \times n}$  its incidence matrix. A previous corollary implies that  $\mathbf{A}$  has rank  $r = n - 1 = n - \dim \ker \mathbf{A}$ .

On the other hand, the above theorem tells us that  $\ell := \dim \text{coker } \mathbf{A}$  equals the number of independent circuits in  $D$ .

Applying the FToLA to  $\mathbf{A}^T$ , we get:  $m = \text{rank}(\mathbf{A}^T) + \dim \ker(\mathbf{A}^T) = r + \ell$  or  $r = m - \ell$ .

Equating the two formulas for rank  $r$ , we obtain:  $r = n - 1 = m - \ell$ , or  $n + \ell = m + 1$  (Euler's Formula for Graphs). Therefore...

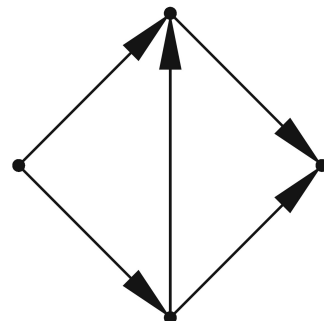
**Theorem:** If  $G$  is a connected graph, then:  $(\# \text{ vertices}) + (\# \text{ independent circuits}) = (\# \text{ edges}) + 1$ .

**Example:** Write out an incidence matrix  $\mathbf{A}$  of the digraph depicted.

Find a basis of the cokernel of the incidence matrix, and its dimension.

What does this tell you about the number of independent circuits in the digraph?

How does this compare with Euler's formula?



We label the order of the edges/vertices by going top to bottom, and breaking ties by going left to right.

So, we start out w/1 as the top vertex, then the two horizontal ones are 2 and 3, with 4 at the bottom.

The upper left edge is 1, the center edge is 2, the upper right is 3, the lower left is 4, with the lower right being 5. Therefore:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

$$\mathbf{A}^T = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables:  $c_4 = s$ ,  $c_5 = t$ . Therefore:  $c_3 = -t$ ,  $c_2 = s - t$ ,  $c_1 = -(s - t) - c_5 = -s$ .

So,  $(-s, s - t, -t, s, t) = s(-1, 1, 0, 1, 0) + t(0, -1, -1, 0, 1)$ ,

And  $\ker \mathbf{A}^T = \text{coker } \mathbf{A} = \text{span}\{(-1, 1, 0, 1, 0), (0, -1, -1, 0, 1)\}$ .

$\dim(\text{coker } \mathbf{A}) = 2$ .

Two independent circuits in the digraph.