

Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

2.5 The Fundamental Matrix Subspaces

Definition: The **image** (or **column space** or **range**) of an $m \times n$ matrix $\mathbf{A}$ is the subspace:

$\text{img } \mathbf{A} \subset \mathbb{R}^m$ spanned by its columns.

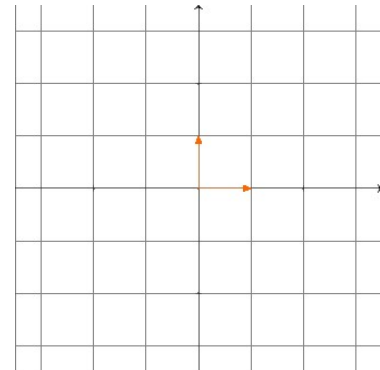
$$\text{img } \mathbf{A} = \text{span}\{\text{cols } \mathbf{A}\} \subseteq \mathbb{R}^m$$

$$= \left\{ \vec{b} \in \mathbb{R}^m \mid \vec{b} = a_1 \vec{c}_1 + \dots + a_n \vec{c}_n \text{ where } a_i \in \mathbb{R}, \text{ and } \vec{c}_i \text{ are columns of } \mathbf{A} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^m \mid \mathbf{A} \vec{c} = \vec{b} \text{ has a soln} \right\}.$$

Definition: The *kernel* of $\mathbf{A}$ is the subspace (denoted $\ker \mathbf{A} \subset \mathbb{R}^n$) consisting of all

vectors that are *annihilated* by $\mathbf{A}$, so $\ker \mathbf{A} = \left\{ \vec{z} \in \mathbb{R}^n \mid \mathbf{A} \vec{z} = \vec{0} \right\} \subset \mathbb{R}^n$.



See animation in lecture.

Remark: If $\mathbf{A}$ is an $m \times n$ matrix: $\ker \mathbf{A}$ lives in $\mathbb{R}^n$ and $\text{img } \mathbf{A}$ lives in $\mathbb{R}^m$.



Example: $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$ ($m = 2, n = 3$) (image?)

$$\text{img } \mathbf{A} = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}. \quad (??)$$

Note: This is not a basis, as columns are dependent:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad (\text{in fact, all columns are parallel}) \quad (\text{find basis?})$$

So: $\text{img } \mathbf{A}$ is 1D, with basis $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

More generally, rather than inspecting the columns for dependencies by hand, we can row reduce $\mathbf{A}$.

The pivot columns identify which columns of the original matrix $\mathbf{A}$ form a basis for $\text{img } \mathbf{A}$.

Example (continued): $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} =: \mathbf{U}.$

This tells us the first column of the original matrix $\mathbf{A}$ is all we need.

So: $\dim \operatorname{img} \mathbf{A} = \# \text{ cols with pivots} = \operatorname{rank} \mathbf{A}.$

! *CAUTION:* The columns of $\mathbf{U}$ with pivots are *NOT* a basis for $\operatorname{img} \mathbf{A}$, and usually don't belong to $\operatorname{img} \mathbf{A}$.

For example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the previous example is not in $\operatorname{img} \mathbf{A} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}.$

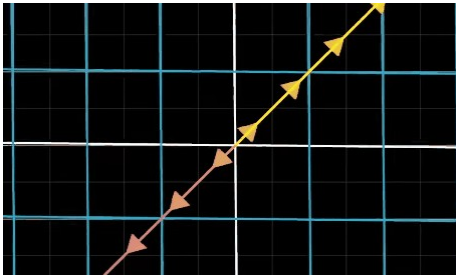
Observe: Number of cols w/pivots is $\dim \operatorname{img} \mathbf{A}.$
 Number of cols w/out pivots is $\dim \ker \mathbf{A}.$

Conclude:

Rank-Nullity Thm: Let $\mathbf{A}$ be an $m \times n$ matrix. Then $n = \dim \operatorname{img} \mathbf{A} + \dim \ker \mathbf{A} = \operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A}.$

Intuitively: the input vector is n -dimensional.
 Each of n dimensions is either annihilated (goes to $\vec{0}$), or survives to the image in $\mathbb{R}^m.$

Examples: $\mathbf{A}^{5 \times 7}$ of rank 3 has $\dim \ker \mathbf{A} = ??$
 $= 4.$



[see animation in class]

$\mathbf{A}^{6 \times 8}$ with $\dim \ker \mathbf{A} = 2$ has rank 6.

Example: $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ -2 & 4 & -6 \\ 3 & 0 & -1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 6 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So first and second cols have pivots.

Thus, $\begin{bmatrix} 1 \\ -3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$ is a basis for $\operatorname{img} \mathbf{A}$, which is a 2D subspace of $\mathbb{R}^4.$

$\mathbf{A}$ has a 1D kernel of $\mathbb{R}^3.$

Finding Sols to Nonhomogeneous Systems

Fix $\mathbf{A}$ and $\vec{b}$, and suppose there are two sols: $\mathbf{A}\vec{x} = \vec{b}$ and $\mathbf{A}\vec{y} = \vec{b}$.

Then their difference $\vec{z} = \vec{x} - \vec{y}$ satisfies: $\mathbf{A}\vec{z} = \mathbf{A}(\vec{x} - \vec{y}) = \mathbf{A}\vec{x} - \mathbf{A}\vec{y} = \vec{b} - \vec{b} = \vec{0}$.

Conclude: The difference of any two sols $\vec{z}$ is in the kernel of $\mathbf{A}$.

And, every soln $\vec{x}$ can be written as $\vec{x} = \vec{y} + \vec{z}$, for any given soln $\vec{y}$ and an element $\vec{z}$ of the kernel.

How? For any given soln $\vec{y}$ let's find any $\vec{x}$. First, subtract them: $\vec{x} - \vec{y} = \vec{z} \in \ker \mathbf{A}$. Now solve for $\vec{x}$.

Thus, if we know:

- $\ker \mathbf{A}$, and
- a single soln $\vec{x}^*$ to $\mathbf{A}\vec{x} = \vec{b}$,

then we can find *all* sols.



Theorem (nonhomogeneous sols): Suppose x^* is a soln to $\mathbf{A}\vec{x} = \vec{b}$. Then, all *other* sols are of the form: $\vec{x}^* + \vec{z}$, where $\vec{z} \in \ker \mathbf{A}$.

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$ and $\vec{b} := \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. Find *all* sols to $\mathbf{A}\vec{x} = \vec{b}$.

Row reduce $[\mathbf{A}|\vec{b}]$ to get: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Note c_3 is free. So, if we set $c_3 = 0$, we find $c_1 = 3$ and $c_2 = 1$.

So, $x^* = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is one soln.

Next, find kernel. General soln to $\mathbf{A}\vec{x} = \vec{0}$.

$$x_3 = s, \quad x_2 = 2s, \quad x_1 = 1s, \text{ so}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}. \quad \text{So, } \ker \mathbf{A} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Thus, *all* sols to $\mathbf{A}\vec{x} = \vec{b}$ are of the form $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. This is the "**general soln.**"

Example: Let $\vec{y} := \begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$ and $\vec{y} \cdot \vec{x} = 6$. Find a general soln.

Due to a property of the dot product:

$$\ker \vec{y} = \{ \text{vectors perp. to } \begin{bmatrix} 2 & -1 & 5 \end{bmatrix} \} = \{ \text{plane defined by } 2x - y + 5z = 0 \}$$

Observe $\vec{y}$ is already "reduced" with c_2, c_3 free.

$$c_2 = s, \quad c_3 = t, \quad c_1 = \frac{1}{2}s - \frac{5}{2}t. \quad (c_1, c_2, c_3) = \left(\frac{1}{2}s - \frac{5}{2}t, \quad s, \quad t \right)$$

$$\ker \vec{y} = \text{span} \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

One soln to $\vec{y} \cdot \vec{x} = 6$ is ??

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

So, general soln is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}.$

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $\vec{b} := \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Find the general soln.

Row reduce: $\left[\mathbf{A} | \vec{b} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 7 & 1 & -7 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{7} & 1 \\ 0 & 1 & \frac{1}{7} & -1 \end{array} \right]. \quad (\text{one soln?})$

So, c_3 free. If $c_3 = 0$, then $c_1 = 1$ and $c_2 = -1$.

So one soln is: $\vec{x}^* = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Next, find $\ker \mathbf{A}$, and the general soln to $\mathbf{A}\vec{x} = \vec{0}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}. \text{ So, } \ker \mathbf{A} = \text{span}\left(\left(-\frac{2}{7}, -\frac{1}{7}, 1\right)\right)$$

Thus, general soln to $\mathbf{A}\vec{c} = \vec{b}$ is: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}$.

Example: Characterize the kernel of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$.

Example: Characterize the image and kernel of $\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 1 & 2 & -3 & 1 \end{bmatrix}$.

Proposition (General): Given $\mathbf{A}^{m \times n}$, the following conditions are equivalent:

- ♦ $\ker \mathbf{A} = \{\vec{0}\}$, i.e., the homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ has the unique soln $\vec{x} = \vec{0}$.
- ♦ $\text{rank } \mathbf{A} = n$.
- ♦ $\mathbf{A}\vec{x} = \vec{b}$ has no free variables.
- ♦ $\mathbf{A}\vec{x} = \vec{b}$ has a unique soln for each $\vec{b} \in \text{img } \mathbf{A}$.

Proposition (Square): Given $\mathbf{A}^{n \times n}_{(\text{square})}$, the following conditions are equivalent:

- ♦ $\ker \mathbf{A} = \{\vec{0}\}$
- ♦ $\text{rank } \mathbf{A} = n$.
- ♦ $\mathbf{A}$ is nonsingular.
- ♦ $\text{img } \mathbf{A} = \mathbb{R}^n$.

The Superposition Principal

For homogeneous systems $\mathbf{A}\vec{x} = \vec{0}$, superposition allows one to generate new sols by combining known sols.

For inhomogeneous systems $\mathbf{A}\vec{x} = \vec{b}$, superposition combines the sols corresponding to different in-homogeneities $\vec{b}$.

In physical applications, the in-homogeneities typically represent external forces $\vec{f}$, and sols $\vec{x}$ represent the responses of the physical apparatus.



hulk smash

The linear superposition principle says that if we know how the system responds to the *individual* forces ($\vec{f}$, $\vec{g}$, etc.), we immediately know it's response to any combination thereof.

Suppose $\vec{x}_1^*$, $\vec{x}_2^*$ are sols to two in-homogeneous systems, $\mathbf{A}\vec{x} = \vec{b}_1$, and $\mathbf{A}\vec{x} = \vec{b}_2$, respectively (w/the same coefficient matrix $\mathbf{A}$).

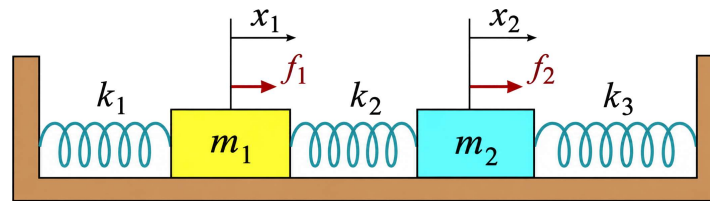
Consider the system $\mathbf{A}\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$, where the RHS is a superposition of the previous two.

Then a particular soln to the combined system is given by the *same superposition* of the previous sols: $\vec{x}^* = c_1\vec{x}_1^* + c_2\vec{x}_2^*$.

...



Proof: $\mathbf{A}\vec{x}^* = \mathbf{A}(c_1\vec{x}_1^* + c_2\vec{x}_2^*) = c_1\mathbf{A}\vec{x}_1^* + c_2\mathbf{A}\vec{x}_2^* = c_1\vec{b}_1 + c_2\vec{b}_2$. ■



Spring-Mass Set-up

Example: The system:
$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$
 models the mechanical response of a pair of masses connected by springs, subject to external forcing $\vec{f} \in \mathbb{R}^2$ (constant acceleration).

Soln $\vec{x} = (x_1, x_2)$ represents displacements of masses, while the entries of $\vec{f} = (f_1, f_2)$ are applied forces.

Assume we can determine that the system responds to a unit force of $\hat{e}_1 = (1, 0)$ on the first mass w/soln $\vec{x}_1^* = (\frac{4}{15}, \frac{1}{15})$, and responds to a unit force of $\hat{e}_2 = (0, 1)$ on the second mass w/soln $\vec{x}_2^* = (\frac{1}{15}, \frac{4}{15})$.

Superposition now gives us the response $\vec{x}$ of the system to *any* general force $\vec{f}$. Observe:

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1\hat{e}_1 + f_2\hat{e}_2, \text{ and hence the soln is:}$$

$$\vec{x} = f_1\vec{x}_1^* + f_2\vec{x}_2^* = f_1 \begin{bmatrix} \frac{4}{15} \\ \frac{1}{15} \end{bmatrix} + f_2 \begin{bmatrix} \frac{1}{15} \\ \frac{4}{15} \end{bmatrix} = \begin{bmatrix} \frac{4}{15}f_1 + \frac{1}{15}f_2 \\ \frac{1}{15}f_1 + \frac{4}{15}f_2 \end{bmatrix}.$$

Thm (Superposition): Suppose that $\vec{x}_1^*, \dots, \vec{x}_k^*$ are particular sols to each of the in-homogeneous linear systems

$$\mathbf{A}\vec{x} = \vec{b}_1, \mathbf{A}\vec{x} = \vec{b}_2, \dots \mathbf{A}\vec{x} = \vec{b}_k. \text{ Then, for any choice of scalars } c_1, \dots, c_k,$$

$$\text{a particular soln } \vec{x}^* \text{ to the combined system } \mathbf{A}\vec{x} = c_1\vec{b}_1 + \dots + c_k\vec{b}_k \quad (*)$$

is the superposition: $\vec{x}^* = c_1\vec{x}_1^* + \dots + c_k\vec{x}_k^*$ of individual sols. The general soln to $(*)$ is

$$\vec{x} = \vec{x}^* + \vec{z} = c_1\vec{x}_1^* + \dots + c_k\vec{x}_k^* + \vec{z}, \text{ where } \vec{z} \in \ker \mathbf{A} \text{ is the general soln to the homogeneous system } \mathbf{A}\vec{z} = \vec{0}.$$

Therefore, if we know particular sols $\vec{x}_1^*, \dots, \vec{x}_m^*$ to $\mathbf{A}\vec{x} = \hat{e}_i$ for each $i = 1, \dots, m$, then we can reconstruct a

$$\text{particular soln } \vec{x}^* \text{ to the general linear system } \mathbf{A}\vec{x} = \vec{b}, \text{ by first writing: } \vec{b} := b_1\hat{e}_1 + \dots + b_m\hat{e}_m$$

$$\text{as a linear combination of the basis vectors, and then using superposition to form } \vec{x}^* = b_1\vec{x}_1^* + \dots + b_m\vec{x}_m^*.$$

Example: Find a soln $\vec{x}_1^*$ to the system $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and a soln $\vec{x}_2^*$ to $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Express the soln to $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{x}_1^*$ and $\vec{x}_2^*$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 10 & 3 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{10} & \frac{1}{10} \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{10} & -\frac{2}{10} \\ 0 & 1 & \frac{3}{10} & \frac{1}{10} \end{array} \right].$$

$$\text{So: } \vec{x}_1^* = \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \end{bmatrix} \text{ and } \vec{x}_2^* = \begin{bmatrix} -\frac{2}{10} \\ \frac{1}{10} \end{bmatrix}.$$

Observe that $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, the particular soln we are seeking is

$$\vec{x}_3^* = 1\vec{x}_1^* + 4\vec{x}_2^* = \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \end{bmatrix} + 4 \begin{bmatrix} -\frac{2}{10} \\ \frac{1}{10} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{7}{10} \end{bmatrix}. \text{ And to check our work, observe that:}$$

$$\mathbf{A}\vec{x}_3^* = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -\frac{2}{5} \\ \frac{7}{10} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \quad \checkmark$$

Adjoint Systems, Cokernel, and Coimage

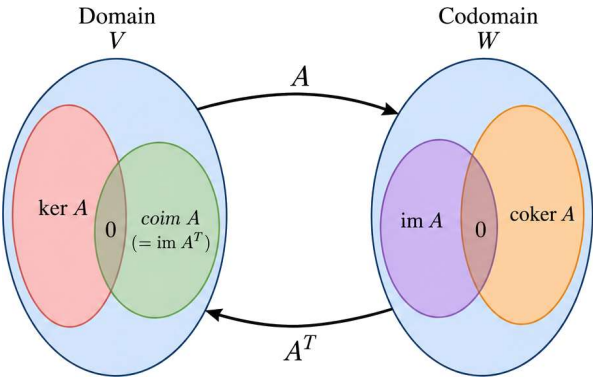
Definition: The *adjoint* to the system $\mathbf{A}\vec{x} = \vec{b}$, of m eqns in n unknowns, is $\mathbf{A}^T\vec{y} = \vec{f}$ consisting of n eqns in m unknowns $\vec{y} \in \mathbb{R}^m$ w/RHS $\vec{f} \in \mathbb{R}^n$.

On the surface, there appears to be no direct connection between the soln sets of a linear system and its adjoint. However, the two are linked.

Definitions: The *coimage* of an $m \times n$ matrix $\mathbf{A}$ is the image of its transpose, $coimg \mathbf{A} = img \mathbf{A}^T = \{ \mathbf{A}^T\vec{y} \mid \vec{y} \in \mathbb{R}^m \} \subset \mathbb{R}^n$.

The coimage coincides with the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$, called $\mathbf{A}$'s *row space*.

The *cokernel* of $\mathbf{A}$ is the kernel of its transpose, $coker \mathbf{A} = \ker \mathbf{A}^T = \{ \vec{w} \in \mathbb{R}^m \mid \mathbf{A}^T\vec{w} = \vec{0} \} \subset \mathbb{R}^m$, that is, the set of sols to the homogeneous adjoint system.



The adjoint system $\mathbf{A}^T\vec{y} = \vec{f}$ has a soln **iff** $\vec{f} \in img \mathbf{A}^T = coimg \mathbf{A}$.

Observe that if $\mathbf{A}^T\vec{w} = \vec{0}$, then taking the transpose of both sides:

$$(\mathbf{A}^T\vec{w})^T = \vec{0}^T \Rightarrow \vec{w}^T \mathbf{A} = \vec{0}^T.$$

If we let $\vec{r} := \vec{w}^T$, we can define the following.

Definition: The cokernel of $\mathbf{A}$ can be identified w/the set of all row vectors $\vec{r}$ satisfying $\vec{r}\mathbf{A} = \vec{0}^T$, and therefore the cokernel is referred to as the *left null space* of $\mathbf{A}$.

Example: Find the *general* soln to the adjoint of $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{bmatrix}$.

First: $\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{bmatrix}$. Then solve the system $\mathbf{A}^T\vec{y} = \vec{f}$.

$$\begin{bmatrix} 1 & 0 & 1 & f_1 \\ -3 & 1 & -2 & f_2 \\ -7 & 5 & -2 & f_3 \\ 9 & -3 & 6 & f_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & f_1 \\ 0 & 1 & 1 & 3f_1 + f_2 \\ 0 & 5 & 5 & 7f_1 + f_3 \\ 0 & 0 & 0 & 3f_2 + f_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & f_1 \\ 0 & 1 & 1 & 3f_1 + f_2 \\ 0 & 0 & 0 & -8f_1 - 5f_2 + f_3 \\ 0 & 0 & 0 & 3f_2 + f_4 \end{bmatrix}.$$

This requires $f_4 = -3f_2$ and $f_3 = 8f_1 + 5f_2$. We have one free column: $y_3 = s$.

$$\text{Satisfying these yields: } \vec{y} = \begin{bmatrix} f_1 - s \\ 3f_1 + f_2 - s \\ s \end{bmatrix} = \begin{bmatrix} f_1 \\ 3f_1 + f_2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

First term on the RHS represents a particular soln,

the second is the general element of the cokernel: $\ker \mathbf{A}^T = \text{co} \ker \mathbf{A}$.

The Fundamental Theorem of Linear Algebra

Observe that the rank of a matrix r (# of pivots), indicates # of independent columns, but also # of independent rows! Therefore:

Thm (FToLA): Given $\mathbf{A}^{m \times n}$, let r be its rank. Then, $\dim \text{co} \text{img} \mathbf{A} = \dim \text{img} \mathbf{A} = \text{rank} \mathbf{A} = \text{rank} \mathbf{A}^T = r$, thus

$$\dim \ker \mathbf{A} = n - r, \quad \dim \text{co} \ker \mathbf{A} = m - r. \quad [\text{Proof in book.}]$$

$\mathbf{A}$ and $\mathbf{A}^T$ have same rank, even though their row echelon forms are quite different and almost never transposes of each other.

Basis for Subspaces

Given $\mathbf{A}^{m \times n}$ w/row echelon form $\mathbf{U}$, to find a basis for:

- ♦ $\text{img} \mathbf{A}$: choose the r columns of $\mathbf{A}$ in which the pivots appear in $\mathbf{U}$ (col space);
- ♦ $\ker \mathbf{A}$: write the gen. soln to $\mathbf{A}\vec{x} = \vec{0}$ as a linear combo of $n - r$ basis vectors whose coefficients are the free variable params;
- ♦ $\text{co} \text{img} \mathbf{A}$: choose the r nonzero rows of $\mathbf{U}$ (row space);
- ♦ $\text{co} \ker \mathbf{A}$: write the gen. soln to adjoint $\mathbf{A}^T \vec{y} = \vec{0}$ as a linear combo of $m - r$ basis vectors whose coefficients are the free variable params.

Example: Using the following vectors as the columns of $\mathbf{A}$, find the dimension of, and a basis for, each of the four fundamental subspaces: $\text{img} \mathbf{A}$, $\ker \mathbf{A}$, $\text{co} \text{img} \mathbf{A}$, and $\text{co} \ker \mathbf{A}$.

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Image: Three pivot points imply the image is 3D, spanned by the first, second, and fourth columns of $\mathbf{A}$.

Therefore, a basis is: $\mathcal{B}_{img} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$.

Here $m = 4$, $n = 5$, and $r = 3$. Therefore, $\dim \ker \mathbf{A} = 5 - 3 = 2$, $\dim \text{coimg } \mathbf{A} = 3$, and $\dim \text{coker } \mathbf{A} = 4 - 3 = 1$.

Kernel: Solve $\mathbf{A}\vec{x} = \vec{0}$ using the row echelon form above:

$$x_1 - 3x_3 + x_4 + 2x_5 = 0, \quad x_2 - 4x_3 - 3x_4 + x_5 = 0, \quad x_4 - 2x_5 = 0.$$

The free variables are $x_3 = s$ and $x_5 = t$. Then $x_4 = 2t$, $x_1 = 3s - 4t$, and $x_2 = 4s + 5t$.

$$\vec{x} = \begin{bmatrix} 3s - 4t \\ 4s + 5t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, a basis is $\mathcal{B}_{ker} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, and $\dim \ker \mathbf{A} = 2$.

Coimage: The three nonzero rows of the row echelon form give a basis for the row space:

$$\mathcal{B}_{coimg} = \{(1, 0, -3, 1, 2), (0, 1, -4, -3, 1), (0, 0, 0, 1, -2)\}.$$

Thus $\dim \text{coimg } \mathbf{A} = 3$. These are vectors in $\mathbb{R}^5$, written as rows.

Cokernel: Solve the adjoint system $\mathbf{A}^T \vec{y} = \vec{0}$:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -3 \\ -3 & -4 & 1 & 6 \\ 1 & -3 & -8 & 7 \\ 2 & 1 & -6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $y_4 = u$. Then $y_1 = 10u$, $y_2 = -5u$, and $y_3 = 4u$, so

$$\vec{y} = u \begin{bmatrix} 10 \\ -5 \\ 4 \\ 1 \end{bmatrix}.$$

Therefore, a basis is $\mathcal{B}_{\text{co ker}} = \left\{ \begin{bmatrix} 10 \\ -5 \\ 4 \\ 1 \end{bmatrix} \right\}$, and $\dim \text{co ker } \mathbf{A} = 1$.

Checking dimensions: $\dim \text{coimg } \mathbf{A} + \dim \text{ker } \mathbf{A} = 3 + 2 = 5 = n$, and
 $\dim \text{img } \mathbf{A} + \dim \text{co ker } \mathbf{A} = 3 + 1 = 4 = m$.