

Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

2.4 Basis and Dimension

A **basis** combines the ideas of **spanning** and **linear independence**.

It gives us enough vectors to describe the whole space, without any redundancy.

This leads naturally to **dimension**: how many independent directions does the space have?

Definition: A basis $\mathcal{B}$ of a vector space V is a finite collection of elements $\vec{v}_1, \dots, \vec{v}_n \in V$ that

- **spans** V , and
- is **linearly independent**.

Example: $\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. But so is ...

$$\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}, \text{ as is } \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}, \text{ etc.}$$

?



How many vectors does a basis have? And how can we test some vectors to confirm they are a basis?

Theorem: Every basis $\mathcal{B}$ of $\mathbb{R}^n$ consists of exactly n vectors. Furthermore, a set of n vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ is a basis **iff** the $n \times n$ matrix $\mathbf{A} = (\vec{v}_1 \dots \vec{v}_n)$ is nonsingular; in other words $\text{rank } \mathbf{A} = n$.

Proof: [Exactly n] Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of $\mathbb{R}^n$, and let $\mathbf{A} = (\vec{v}_1 \dots \vec{v}_k)$. Since the vectors span $\mathbb{R}^n$, $\mathbf{A}\vec{x} = \vec{b}$ must be solvable for every $\vec{b} \in \mathbb{R}^n$, so there must be a pivot in every row; hence $k \geq n$.

Next, since the vectors are linearly independent, $\mathbf{A}\vec{x} = \vec{0}$ has only the trivial soln.

But if $k > n$, our previous thm says a homogeneous system w/more unknowns than eqns has a nontrivial soln.

Thus $k \leq n$, and therefore $k = n$.

[basis $\Leftrightarrow$ nonsingular] For n vectors in $\mathbb{R}^n$, $\mathbf{A}$ is square. By definition, the vectors form a basis **iff** they span $\mathbb{R}^n$ and

are linearly independent, which occurs **iff** $\mathbf{A}$ has n pivots, i.e. $\text{rank } \mathbf{A} = n$.

(Columns span $\mathbb{R}^n$ **iff** pivot in every row. Columns are indep. **iff** pivot in every column.)

Sub-Explanation: Columns span $\mathbb{R}^n$ **iff** pivot in every row.

A pivot in every row means $\mathbf{A}\vec{x} = \vec{b}$ is solvable for every $\vec{b} \in \mathbb{R}^n$. But $\mathbf{A}\vec{x} = \vec{b}$ says precisely that $\vec{b}$ is a linear combination of the columns of $\mathbf{A}$ (so they span $\mathbb{R}^n$).

Columns are indep. **iff** pivot in every column.

Every column has a pivot means $\mathbf{A}\vec{x} = \vec{0}$ has no free variables, which means its only soln is $\vec{x} = \vec{0}$.

That is exactly the definition of linear independence of the columns.

By a previous result, $\text{rank } \mathbf{A} = n$ **iff** $\mathbf{A}$ is nonsingular. ■

Now we have a way of identifying a basis in $\mathbb{R}^n$! Check if $\mathbf{A} = (\vec{v}_1 \dots \vec{v}_n)$ is nonsingular.

But what if we are in a more general linear space, not $\mathbb{R}^n$?

Theorem/Definition: Suppose vector/linear space V has a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ for some $n \in \mathbb{N}$. Then every other basis of V has n elements in it. This number is called the **dimension of V** , and is written $\dim V = n$.

The proof of this theorem rests on the following lemma:

Lemma: Suppose $\vec{v}_1, \dots, \vec{v}_n$ span a linear space V . Then every set of $k > n$ elements $\vec{w}_1, \dots, \vec{w}_k \in V$ is linearly dependent.

Proof of Lemma: We can write each element $\vec{w}_j = a_{1j}\vec{v}_1 + a_{2j}\vec{v}_2 + \dots + a_{nj}\vec{v}_n$ (where $j = 1, \dots, k$), as a linear combo of

the spanning set. Then, $c_1\vec{w}_1 + \dots + c_k\vec{w}_k = c_1(a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \dots + a_{n1}\vec{v}_n) + \dots + c_k(a_{1k}\vec{v}_1 + a_{2k}\vec{v}_2 + \dots + a_{nk}\vec{v}_n)$
 $= (c_1a_{11} + \dots + c_ka_{1k})\vec{v}_1 + \dots + (c_1a_{n1} + \dots + c_ka_{nk})\vec{v}_n.$ (collected the $\vec{v}_i$)

To prove the lemma, it is sufficient to show that $c_1\vec{w}_1 + \dots + c_k\vec{w}_k = \vec{0}$ has a nontrivial $\vec{c} = (c_1, \dots, c_k)$ soln.

Looking term-wise at the "collected" eqn, each of the n terms' coefficients will be zero when

$c_1a_{i1} + \dots + c_ka_{ik} = 0$ (where $i = 1, \dots, n$). Observe this consists of n eqns in $k > n$ unknowns c_j .

A previous thm guarantees that every homogeneous system w/more unknowns than eqns always has a nontrivial soln $\vec{c} \neq \vec{0}$, and this immediately implies that $\vec{w}_1, \dots, \vec{w}_k$ are linearly dependent. ■

Proof of the preceding thm (every basis has same # of elements):

Recall: $\vec{v}_1, \dots, \vec{v}_m$ in V form a basis of V if they span V and are linearly independent.

Let $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ be bases of V .

Since $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent and $\vec{w}_1, \dots, \vec{w}_q$ span V , we have $p \leq q$, by the previous lemma.

Likewise, since $\vec{w}_1, \dots, \vec{w}_q$ are linearly independent and $\vec{v}_1, \dots, \vec{v}_p$ span V , we have $q \leq p$. Therefore, $p = q$. ■

Examples - Find a span, basis, dimension of the following:

♦ $\{y = x\} \subset \mathbb{R}^2$...

Span is the line $y = x$. Written: $\text{span}\{(1, 1)\}$, $\mathcal{B} = \{(1, 1)\}$, $\dim(A) = 1$.

♦ Any line in $\mathbb{R}^2$ through $\vec{0}$

$\text{span}\{\vec{u}\}$, for some $\vec{u} \neq \vec{0}$ on the line, $\mathcal{B} = \{\vec{u}\}$, $\dim(B) = 1$.

♦ Any plane in $\mathbb{R}^3$ through $\vec{0}$

Spanned by two indep. vecs $\vec{v}_1, \vec{v}_2$ in the plane, which also forms basis $\{\vec{v}_1, \vec{v}_2\} \Rightarrow \dim(C) = 2$.

$$\begin{aligned} \text{♦ } \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 42 \\ 0 \end{bmatrix}\right\} &= \dots \\ &= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \dim(D) = 2. \end{aligned}$$

♦ $\mathbb{R}^n$...

$\text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, $\dim(\mathbb{R}^n) = n$.

Suppose V is an n -dimensional vector space. Then you should be able to convince yourself that:

- Every set of more than n elements of V is linearly dependent.
- No set of fewer than n elements spans V .
- A set of n elements forms a basis **iff** it spans V
- A set of n elements forms a basis **iff** it is linearly independent.

❗ A basis does more than span the space, it gives each vector a unique set of coordinates.

Lemma: The elements $\vec{v}_1, \dots, \vec{v}_n$ form a basis of V **iff** every $\vec{x} \in V$ can be written *uniquely* as a linear combination of the basis elements: $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \sum_{i=1}^n c_i \vec{v}_i$.

Pf ($\Rightarrow$): The fact that a basis spans V implies that every $\vec{x} \in V$ can be written as some linear combo of the basis elements.

$$\text{Suppose we write an element } \vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = d_1 \vec{v}_1 + \dots + d_n \vec{v}_n \quad (*)$$

as two *different* combinations. Subtracting one from the other should give us the zero vector: $\vec{x} - \vec{x} = \vec{0}$.

We obtain: $(c_1 - d_1) \vec{v}_1 + \dots + (c_n - d_n) \vec{v}_n = \vec{0}$. The LHS is a linear combination of the linearly independent

basis elements, and hence vanishes **iff** all of its coefficients $c_i - d_i = 0$,

meaning that the two linear combinations $(*)$ are one and the same. ■