

Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

2.3 Span and Linear Independence

Two fundamental questions arise when working with a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$:

- What can we build from them, and
- Do we need all of them?

The *Span* of these vectors describes every vector we can build from them by using "*linear combinations*," while *linear independence* tells us whether any of our vectors are redundant.

Definition: Let $\vec{v}_1, \dots, \vec{v}_k$ be elements of a vector space V . A *linear combination* of the elements $\vec{v}_1, \dots, \vec{v}_k$ is a sum the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \sum_{i=1}^k c_i\vec{v}_i$, where the coefficients $c_1, \dots, c_k$ are any scalars. Their *span* is the subset $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$ consisting of all possible linear combinations with scalars $c_1, \dots, c_k \in \mathbb{R}$.

Example: Observe that with $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ you can get anywhere in the Cartesian coordinate plane.

If you wish to go to (a, b) , use the linear combination $a\vec{e}_1 + b\vec{e}_2 = (a, 0) + (0, b) = (a, b)$.

Observe that if you were missing either $\vec{e}_1$ or $\vec{e}_2$, you could no longer get everywhere.

Also note that if you had any 3rd vector (a, b) , it would be redundant. Why?

We showed above that we could get to that 3rd vector using $\vec{e}_1$ and $\vec{e}_2$.

So, by the definition above, $\text{span}\{\vec{e}_1, \vec{e}_2\}$ is the whole Cartesian coordinate plane.

And it turns out that spans always generate subspaces!

Proposition: The span $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ of any finite collection of vector space elements $\vec{v}_1, \dots, \vec{v}_k \in V$ is a subspace of the underlying vector space V .

Proof: Recall $W = \{c_1\vec{v}_1 + \dots + c_m\vec{v}_m \mid c_i \in \mathbb{R}\}$. Must show: W is nonempty and has closure properties.

a) $\vec{0} \in W$? (non-empty?)

Set $c_1 = c_2 = \dots = c_m = 0$. Then, $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$.

b) Closed under addition?

Let $\vec{a}, \vec{b} \in W$. Then, there exist constants a_i, b_i such that:

$$\vec{a} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m,$$

$$\vec{b} = b_1 \vec{v}_1 + \dots + b_m \vec{v}_m.$$

Therefore: $\vec{a} + \vec{b} = (a_1 + b_1) \vec{v}_1 + \dots + (a_m + b_m) \vec{v}_m \in W$. (it's a linear combo of the $\vec{v}_i$)

c) Closed under scalar multiplication?

If $\vec{a} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m$ and $k \in \mathbb{R}$,

Then $k\vec{a} = (ka_1) \vec{v}_1 + \dots + (ka_m) \vec{v}_m \in W$. (it's a linear combo of the $\vec{v}_i$)



Linear Independence and Dependence

Which vector(s) are dead weight? Who can be cut from the team?



Definition: Elements $\vec{v}_1, \dots, \vec{v}_k \in V$ are called *linearly dependent* if there exists scalars $c_1, \dots, c_k$, *not all zero*, such that $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$.

Elements that are not linearly dependent are called *linearly independent*.

❗ If $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, it doesn't mean every $\vec{v}_i$ can be written as a combination of the others.

It implies at least one $\vec{v}_i$ can.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix} \right\} \dots$

$$7\vec{v}_1 + 5\vec{v}_2 + 0\vec{v}_3 = \vec{v}_4. \quad \text{However, } \vec{v}_3 \text{ can't be written in terms of the others. (why?)}$$

Systems of Eqns and Independence

Thm: Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and let $\mathbf{A}^{n \times k} = (\vec{v}_1 \dots \vec{v}_k)$ be the corresponding matrix whose columns are the given vectors.

♦ Vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent **iff** there is a nonzero soln $\vec{c} \neq \vec{0}$ to the homogeneous $\mathbf{A}\vec{c} = \vec{0}$.

♦ Vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent **iff** the only soln to the homogeneous $\mathbf{A}\vec{c} = \vec{0}$ is the trivial: $\vec{c} = \vec{0}$.

♦ Vector $\vec{b}$ lies in the span of $\vec{v}_1, \dots, \vec{v}_k$ **iff** the linear system $\mathbf{A}\vec{x} = \vec{b}$ is compatible (has at least one soln).

We prove the first statement, leaving the other two as exercises for you.

Proof: The condition that $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent is that there exists a nonzero vector

$$\vec{c} = (c_1, c_2, \dots, c_k) \neq \vec{0} \text{ such that } c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}.$$

$$\text{But recall that } \mathbf{A}\vec{c} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k.$$

Therefore, linear dependence is identical to requiring the existence of a nontrivial soln to $\mathbf{A}\vec{c} = \vec{0}$. ■

Lemma: Any collection of $k > n$ vectors in $\mathbb{R}^n$ is linearly dependent. (recall $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$ and dependence w/any additional vector)

Proof (sketch): This comes from the 1st bullet point in the thm, along w/the theorem in §1.8 regarding rank & nontrivial solns of homogeneous systems.

Proposition: A set of k vectors in $\mathbb{R}^p$ is linearly independent **iff** the corresponding $p \times k$ matrix $\mathbf{A}$ has rank k .

In particular, this requires $k \leq p$.

Proof (sketch): This comes from the 2nd bullet point in the thm, along w/the theorem in §1.8.

Proposition: A collection of k vectors spans $\mathbb{R}^n$ **iff** their $n \times k$ matrix has rank n . In particular, this requires $k \geq n$.

Proof (sketch): This comes from the 3rd bullet pt in the thm, along w/an earlier solvability criterion from

Gaussian elimination. Recall: Gaussian elimination tells us $\mathbf{A}\vec{x} = \vec{b}$ has a soln for every $\vec{b} \in \mathbb{R}^n$ **iff**

$\mathbf{A}$ has a pivot in every row, meaning $\text{rank}(\mathbf{A}) = n$.

Definition: A *linear relation* among $\vec{v}_1, \dots, \vec{v}_m$ is a linear combination that produces the zero vector: $c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = \vec{0}$.

Determining dependence of $\{\vec{v}_1, \dots, \vec{v}_m\}$:

Dependent

- ♦ At least one $\vec{v}_i$ is a linear combination of the others.
- ♦ There exists at least one nontrivial **linear relation**: $c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = \vec{0}$

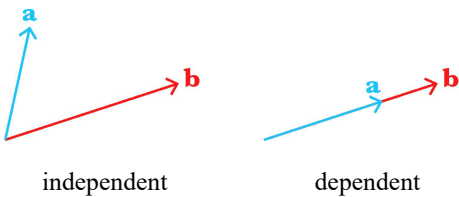
Independent

- ♦ No $\vec{v}_i$ is a combination of others.
- ♦ The only **linear relation** is the trivial one:
 $0\vec{v}_1 + \dots + 0\vec{v}_m = \vec{0}.$

Examples:

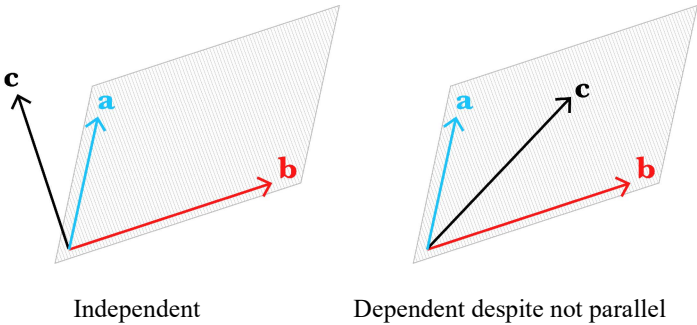
Two vectors in $\mathbb{R}^n$ are linearly...

- ◆ Dependent if they're parallel. (one is a multiple of the other)
- ◆ Independent if not parallel.



Three (or more) vectors? Parallel criteria doesn't work.

- ◆ Why? When none of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are parallel, the vectors may still be dependent (see image).
- ◆ Better criteria: whether one is a linear combo of the other two.



Example: Are $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ independent?

Method 1: Look for possible linear relations:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Observe from first coordinate, c_3 must be 0. This forces second coordinate, c_2 must be 0. And then third coordinate is forced, $c_1 = 0$.

Only linear relation is trivial $\Rightarrow$ independent.

Method 2:
$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \end{array} \right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]. \quad (\text{only trivial soln}) \Rightarrow \text{independent}.$$

Method 3: Add vectors one-at-a-time to check for dependence.

First vector dependent $(0,0,2)$? One vector can't be dependent.

Ensure second vector not multiple of first. $(0,1,3) \neq c(0,0,2)$.

Now check if third vector is a linear combination of the first two.

$$x \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \quad \text{obviously not (due to first component).}$$

No $\vec{v}_i$ is a combination of others? $\Rightarrow$ independent.

Example: Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent.

The corresponding 4×3 matrix is $\mathbf{A} := \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Must show this has rank 3.

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Three pivot pts imply rank 3.}$$

Which of the following vectors are in their span? $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Keep in mind, the span consists of sums $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$, for any choice of $c_i \in \mathbb{R}$.

So the zero vector $\vec{0}$ can be satisfied when the $c_i = 0$.

For the others, you can attempt to solve the system: $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \vec{c} = \vec{b}.$

Observe, we can solve them all *simultaneously* in the following augmented matrix: $[\mathbf{A} \mid \vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$ 

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 3 & -3 & 0 & -2 & 0 \\ 0 & 3 & -3 & 0 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & -3 & -2 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

Observe that $\vec{b}_2$ has no soln due to the last row.

According to this result, $\vec{b}_1$ can be formed by: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = 1\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3.$

And indeed: $1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}. \quad \checkmark$

Similarly, $\vec{b}_3$ can be formed by: $0\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. ✓

Suppose (a, b, c, d) lies in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. What conditions must a, b, c, d satisfy?

Since $k_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, we must solve $\begin{bmatrix} 1 & -2 & 2 & | & a \\ 0 & 3 & -2 & | & b \\ 2 & -1 & 1 & | & c \\ 1 & 1 & -1 & | & d \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 & a \\ 0 & 3 & -2 & b \\ 0 & 0 & -3 & 3c - 3b - 6a \\ 0 & 0 & 0 & -3d + 3c - 3a \end{bmatrix}.$$

The last row implies that $c = a + d$.

That is the only requirement on the a, b, c, d .

However, you can subsequently determine the requirements on the unique soln determined by this vector:

The 3rd row gives us $k_3 = -(a + d) + b + 2a = b + a - d$.

The 2nd row gives us $3k_2 - 2k_3 = b \Rightarrow 3k_2 - 2(b + a - d) = b$

$$\text{or } k_2 = b + \frac{2}{3}a - \frac{2}{3}d.$$

Lastly, from the first row we have: $k_1 - 2k_2 + k_3 = a \Rightarrow k_1 - 2(b + \frac{2}{3}a - \frac{2}{3}d) + (b + a - d) = a$

$$\Rightarrow k_1 = \frac{4}{3}a + b - \frac{1}{3}d.$$

Therefore, every vector (a, b, c, d) such that $c = a + d$, lies in the span, and specifically:

$$(a, b, c, d) = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \left(\frac{4}{3}a + b - \frac{1}{3}d\right)\vec{v}_1 + \left(b + \frac{2}{3}a - \frac{2}{3}d\right)\vec{v}_2 + (b + a - d)\vec{v}_3.$$