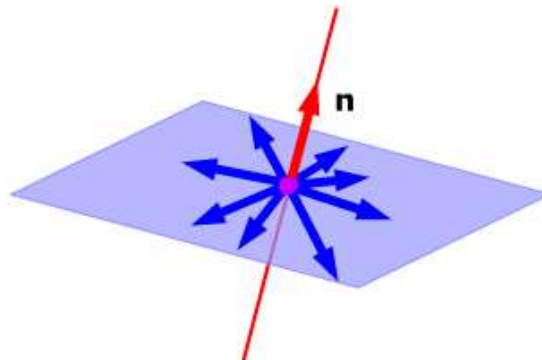


Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

2.2 - Subspaces



Subspaces: Given a vector space V (e.g., $\mathbb{R}^3$ in the image above) and subset W of V (the plane in the image), then W is a subspace if:

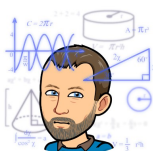
a) W is nonempty (it contains at least one vector),

b) Given $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$, (W is closed under addition)

c) Given $\vec{u} \in W$ and $k \in \mathbb{R}$, we have $k\vec{u} \in W$. (W is closed under scalar multiplication)

And therefore $\vec{0} \in W$. Why?

Also, convince yourself that the x -axis and y -axis (just the axes themselves, no other points), joined together as a subset of $\mathbb{R}^2$, *does not* constitute a subspace.



Does $c \Rightarrow a$?

(no)

Requirement a ensures that $W \neq \emptyset$.

Examples: Which of the following are subspaces?

$A := \{(x, y) \mid x \in [0, 1]\} \subset \mathbb{R}^2$, ...

$B := \{(x, y) \mid y = x\} \subset \mathbb{R}^2$, ...

$C := \{(x, y, 0)\} = \{(x, y, z) \mid z = 0\} \subset \mathbb{R}^3$, ...

$D := \{\vec{0}\} \subset \mathbb{R}^n$, ...

$$E := \{(x, y, 4) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3, \quad \dots$$

More Generally

Theorem: As a result of the properties above, *a subspace is a vector space in its own right*. This new vector space uses the same operations of vector addition, scalar multiplication, and has the same zero element. It satisfies all the vector space properties.

Proof: The proof is immediate. For example, let W be a subspace of V , and let's check **commutativity**.

Subspace elements $\vec{v}, \vec{w} \in W$ can be regarded as elements of V , thus $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ because V is a vector space.

But the subspace closure condition implies that sums also belongs to W , thus commutivity also holds for elements of W .

Establishing the validity of the other properties is equally as easy. ■

Solution Subspace Thm: For $\mathbf{A}^{m \times n}$, the soln set of the homogeneous system $\mathbf{A} \vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof: Let W denote the soln set of the system. Note the trivial soln exists, so W is nonempty.

If $\vec{u}$ and $\vec{v}$ are in W , then $\mathbf{A} \vec{u} = \mathbf{A} \vec{v} = \vec{0}$.

Hence: $\mathbf{A}(\vec{u} + \vec{v}) = \mathbf{A} \vec{u} + \mathbf{A} \vec{v} = \vec{0} + \vec{0} = \vec{0}$.

Thus the sum $\vec{u} + \vec{v}$ is also in W , and hence W is closed under addition.

Next, if $c \in \mathbb{R}$, then $\mathbf{A}(c\vec{u}) = c(\mathbf{A} \vec{u}) = c\vec{0} = \vec{0}$.

Thus $c\vec{u}$ is in W if $\vec{u}$ is in W . Hence W is also closed under scalar multiplication.

Therefore, W is a subspace of $\mathbb{R}^n$. ■

Nonhomogeneous System Sols Thm: The soln set of a nonhomogeneous system $\mathbf{A} \vec{x} = \vec{b}$ is **never** a subspace.

Proof: Let's do proof by contradiction. Let $\vec{u}$ be a soln in W , the set of sols to $\mathbf{A} \vec{x} = \vec{b}$.

And let us make the dubious assumption that W is a subspace.

Let $c = 0 \in \mathbb{R}$. By closure under scalar multiplication, $c\vec{u} = 0\vec{u} = \vec{0}$ is a soln.

Therefore, $\mathbf{A} \vec{0} = \vec{0} = \vec{b}$. (?!)

But we assumed the system was nonhomogeneous: $\vec{b} \neq \vec{0}$.

So we have a contradiction, and our assumption that the set of sols W was a subspace must have been incorrect. ■

Video Tutorial (visually rich and intuitive): https://youtu.be/fNk_zzaMoSs

Exercises

Problem 1 Assume W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$.

Apply the theorems in this section to determine whether or not W is a subspace of $\mathbb{R}^4$.

$$W = \{(3c, 4d, c, d)\}.$$

First, note that the subspace is nonempty since $(3, 4, 1, 1) \in W$, where $c, d = 1$.

We arbitrarily choose two vectors from W by arbitrarily choosing four constants $c_1, d_1, c_2, d_2 \in \mathbb{R}$, giving us $(3c_1, 4d_1, c_1, d_1)$ and $(3c_2, 4d_2, c_2, d_2)$. We then test them for closure under addition:

$$\begin{aligned} & (3c_1, 4d_1, c_1, d_1) + (3c_2, 4d_2, c_2, d_2) \\ &= (3c_1 + 3c_2, 4d_1 + 4d_2, c_1 + c_2, d_1 + d_2) \\ &= (3(c_1 + c_2), 4(d_1 + d_2), c_1 + c_2, d_1 + d_2) \in W \end{aligned}$$

This is because it has the prescribed format $\{(3c, 4d, c, d)\}$, where $c := c_1 + c_2$ and $d := d_1 + d_2$.

Now to test scalar multiplication:

$$\begin{aligned} \alpha(3c_1, 4d_1, c_1, d_1) &= (3\alpha c_1, 4\alpha d_1, \alpha c_1, \alpha d_1) \in W = \{(3c, 4d, c, d)\} \\ &\text{where } c := \alpha c_1 \text{ and } d := \alpha d_1. \end{aligned}$$

Therefore, W is nonempty, closed under addition, and scalar multiplication, and is a subspace of $\mathbb{R}^4$. ■

Problem 2 Reduce the given system to echelon form to find a single soln vector $\vec{u}$ such that the soln space is the set of all scalar multiples of $\vec{u}$.

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0, \\ 2x_1 + 7x_2 + 5x_3 - x_4 &= 0, \\ 2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0. \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{bmatrix} \xRightarrow{\text{trust me}} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus $x_4 = t$ is a parameter (a.k.a. free variable).

We solve for $x_1 = -6t$, $x_2 = 4t$, and $x_3 = -3t$. So,

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-6t, 4t, -3t, t) = t\vec{u}, \text{ where } \vec{u} = (-6, 4, -3, 1).$$

Problem 3 For the following system of eqns, find two solns $\vec{u}$ and $\vec{v}$ such that the soln space is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \xRightarrow{\text{trust me}} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_2 = -s - 3t$, and $x_1 = -s - 5t$. So ...

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-s - 5t, -s - 3t, s, t)$$

$$= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\vec{u} + t\vec{v}, \text{ where } \vec{u} = (-1, -1, 1, 0) \text{ and } \vec{v} = (-5, -3, 0, 1).$$

Problem 4 Let $\mathbf{A}$ be an $n \times n$ matrix, $\vec{b}$ be a nonzero vector, and $\vec{x}_0$ be a soln to the system $\mathbf{A}\vec{x} = \vec{b}$.

Show that $\vec{x}_2$ is another soln **iff** $\vec{x}_2 - \vec{x}_0$ is a soln of the homogeneous system $\mathbf{A}\vec{y} = \vec{0}$.

We are given: $\mathbf{A}\vec{x}_0 = \vec{b}$.

Need to show that: $\mathbf{A}\vec{x}_2 = \vec{b} \Leftrightarrow \mathbf{A}(\vec{x}_2 - \vec{x}_0) = \vec{0}$.

Starting with the left assumption, and trying to show the thing on the right, we have:

$$\mathbf{A}(\vec{x}_2 - \vec{x}_0) = \mathbf{A}\vec{x}_2 - \mathbf{A}\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}. \quad \checkmark$$

Going from right to left, we have: $\mathbf{A}(\vec{x}_2 - \vec{x}_0) = \vec{0} \Rightarrow \mathbf{A}\vec{x}_2 - \mathbf{A}\vec{x}_0 = \vec{0} \Rightarrow \mathbf{A}\vec{x}_2 = \mathbf{A}\vec{x}_0 = \vec{b}$. ■