

Applied Linear Algebra

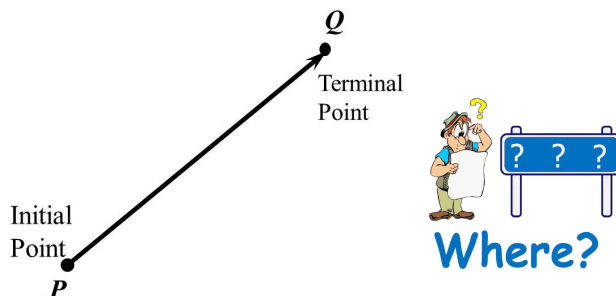
Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

§2.1 - Real Vector Spaces

Vector: a mathematical object consisting of **direction & magnitude**.

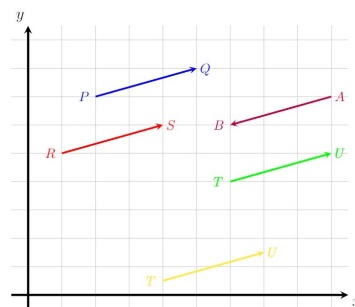
A vector, notated $\vec{v}$, $\mathbf{v}$, or $\overrightarrow{PQ}$, is often drawn as an arrow starting at **initial point** P (tail), and ending at **terminal point** Q (tip/head).

Magnitude: length of arrow.



$\vec{u}, \vec{v}$ are equivalent ($\vec{u} = \vec{v}$) if they share same magnitude & direction, even if they have a different P (and thus a different Q).

We sometimes write $\vec{v} = \langle 2, 1 \rangle$ to mean a vector pointing to $(2, 1)$. But $\langle 2, 1 \rangle$ refers to *any* vector w/same magnitude & direction, located anywhere in $\mathbb{R}^2$.



Zero Vector: $\mathbf{0}$ or $\vec{0}$, has length 0. It's the only vector with no direction.

In 2D (similarly in 3D), if the tail of $\vec{a} = \langle a_1, a_2 \rangle$ is at the origin, then the tip has coordinates (a_1, a_2) .

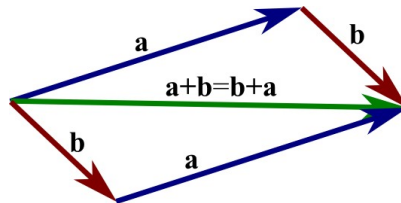
The numbers a_1, a_2 are the **components of $\vec{a}$** .

Vector Spaces

A *vector space* is a set V equipped with w/two *closure* operations:

Vector Addition (V is closed under addition): adding any pair of vectors $\vec{a}, \vec{b} \in V$ produces another vector $\vec{a} + \vec{b} \in V$;

If $\vec{a}, \vec{b}$ are positioned w/the tail of $\vec{b}$ at the tip of $\vec{a}$, then $\vec{a} + \vec{b}$ goes from the tail of $\vec{a}$ to the tip of $\vec{b}$.



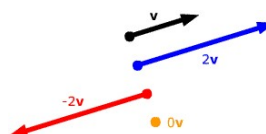
Geometric: Triangle & Parallelogram Law

Scalar Multiplication (V is closed under scalar multiplication):

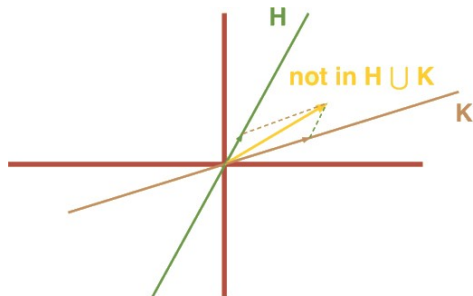
multiplying $\vec{v} \in V$ by $c \in \mathbb{R}$ produces $c\vec{v} \in V$.

$$c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$$

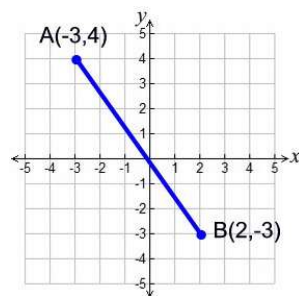
Algebraic



Geometric



Vecs in H and K aren't a v-space. Breaks addition operation



Vecs in line segment aren't a v-space. Breaks scalar operation

To be a vector space, the two closure operations must satisfy the following axioms for all $\vec{u}, \vec{v}, \vec{w} \in V$ and scalars $c, d \in \mathbb{R}$.

Commutativity of Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Associativity of Addition: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

Additive Identity: There is a zero element $\vec{0} \in V$ satisfying $\vec{v} + \vec{0} = \vec{v}$.

Additive Inverse: For each $\vec{v} \in V$, there's an element $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

Distributivity w/Scalars: $(c + d)\vec{v} = c\vec{v} + d\vec{v}$, and $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.

Associativity of Scalar Multiplication: $c(d\vec{v}) = (cd)\vec{v}$.

Scalar Multiplicative Identity: the scalar $1 \in \mathbb{R}$ satisfies $1\vec{v} = \vec{v}$.

Vector spaces are not limited to literal vectors! Example: polynomials of degree 3 or less.

$$P_3(x) := a_0 + a_1x + a_2x^2 + a_3x^3, \text{ for real } a_i.$$

A **vector space**, or more generally a **linear space** is a set of objects with a *reasonable* sense of addition and scalar multiplication (as defined above).

Polynomial spaces satisfy vector space properties: Given p_1, p_2, p_3 in P_3 .

$$\blacklozenge p_1 + p_2 = p_2 + p_1$$

$$\blacklozenge p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$$

$$\blacklozenge p_1 + 0 = 0 + p_1 = p_1$$

$$\blacklozenge p_1 + (-p_1) = (-p_1) + p_1 = 0$$

$$\blacklozenge (c + k)p_1 = cp_1 + kp_1, \text{ for any real } c, k; \text{ and } k(p_1 + p_2) = kp_1 + kp_2, \text{ for any real } k$$

$$\blacklozenge c(kp_1) = (ck)p_1$$

$$\blacklozenge 1(p_1) = p_1$$

The following are elementary consequences of the vector space axioms:

$$\blacklozenge 0\vec{v} = \vec{0},$$

$$\blacklozenge (-1)\vec{v} = -\vec{v},$$

$$\blacklozenge c\vec{0} = \vec{0},$$

$$\blacklozenge \text{ If } c\vec{v} = \vec{0}, \text{ then either } c = 0 \text{ or } \vec{v} = \vec{0}.$$

Function Spaces (Linear Spaces of Functions)

Let $F = \{ \text{all real valued functions on } \mathbb{R} \}$; includes all polynomials, trig functions, exponentials, etc... ...

Observe for $\mathbf{f}, \mathbf{g}$ in F , we have: $(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \in F$ and $(cf)(x) := c(f(x)) \in F$.

The properties of a linear space follow since these functions are real valued. ...

What is the neutral element? $z(x)$...

Linear Space of $m \times n$ Matrices

Example: 2×2 matrices $\mathbf{A} \in \mathbb{R}^{2 \times 2}$. Matrices like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with neutral element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

They are closed under addition and scalar multiplication. Furthermore, given $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and zero matrix $\mathbf{0} \in \mathbb{R}^{2 \times 2}$.

$$\blacklozenge \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\blacklozenge \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$\blacklozenge \mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

$$\blacklozenge \mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{0}$$

$$\blacklozenge (c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}, \text{ for any real } c, k; \text{ and } k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}, \text{ for any real } k$$

$$\blacklozenge c(k\mathbf{A}) = (ck)\mathbf{A}$$

$$\blacklozenge 1(\mathbf{A}) = \mathbf{A}$$

Similar results follow for other sized matrix spaces: $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Infinite Dimensional Linear Spaces



$\mathbb{R}^\infty$, space of vectors $\vec{v} = (a_0, a_1, a_2, \dots)$, where a_i in $\mathbb{R}$. (e.g., decimals $c \in (0, 1)$)

Polynomials: $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where a_i in $\mathbb{R}$, for ALL n in $\{0, 1, 2, \dots\}$.

$5x^4 + 2x - 7$ is $(-7, 2, 0, 0, 5, 0, 0, 0 \dots)$, etc.

Verify that they satisfy the axioms and properties.

Convergent Sequences: Space of all real sequences $(a_0, a_1, a_2, \dots)$ for which $\lim_{n \rightarrow \infty} a_n$ exists as a finite real number.

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$, $(3, 3, 3, 3, \dots)$, $(1, 0, 0, 0, \dots)$, $(0, 1, 0, 0, 0, \dots)$, $(0.9, 0.99, 0.999, \dots)$

Verify that they satisfy the axioms and properties.