

§1.9 - Determinants

A determinant assigns a single scalar to a square matrix. This scalar tells us whether the matrix is invertible and, geometrically, describes the (signed) area or volume scaling of the parallelepiped defined by the matrix's rows.

Determinants are theoretically important in linear algebra and useful in some low dimensional, highly structured problems.

However, for large systems, determinant formulas are usually not a practical way to solve $\mathbf{A}\vec{x} = \vec{b}$.

Elimination and matrix factorizations are more efficient; these also provide a practical way to compute a determinant when one is needed.

Our goal here is to connect determinants with the row operations we already know.

This will give us both a method of computation and a useful test for singularity.

Recall that the determinant of a 2×2 matrix is: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The expression $ad - bc$ vanishes exactly when the 2×2 matrix is singular.

For larger square matrices, we characterize the determinant by how it behaves under row operations, rather than starting with a long formula.

We write either $\det \mathbf{A}$ or $|\mathbf{A}|$; the vertical bars here mean determinant, not absolute value.

Definition: For each square matrix, a *determinant* is a scalar satisfying the following axioms.

1. Adding a multiple of one row to a different row does not change the determinant.
2. Interchanging 2 rows changes the sign of the determinant.
3. Multiplying a row by any scalar (including 0) multiplies the determinant by the same scalar.
4. The determinant of any upper triangular matrix $\mathbf{U}$ is equal to the product of its diagonal entries: $\det \mathbf{U} = u_{11}u_{22}\dots u_{nn}$.

- The first three axioms tell us how to keep track of the determinant during elimination.
- The fourth supplies its value once we reach triangular form; in particular, $\det \mathbf{I} = 1$.
- Scaling one row by a scalar scales the determinant once; scaling the entire $n \times n$ matrix by that scalar scales each of its n rows.

Theorem: For each size n , there exists exactly one determinant function satisfying these axioms.

Thus there exists a well-defined, unique determinant for every square matrix.

There are two claims here:

- existence says that all four rules can be satisfied simultaneously, and
- uniqueness says that they force just one answer for every matrix.

A proof of existence, omitted here, must also establish that the value is independent of the chosen elimination steps.
The elimination argument below establishes uniqueness.

Lemma: Any matrix with one or more all-zero rows has determinant of zero.

Proof: Multiplying an all-zero row by 0 leaves the matrix unchanged.

While axiom 3 multiplies its determinant by 0.

Consequently, its determinant must be (and must have already been) 0. ■

We now use elimination to reduce the calculation to triangular form.

Theorem: If $\mathbf{A} = \mathbf{LU}$ is a regular matrix, then: $\det \mathbf{A} = \det \mathbf{U} = u_{11}u_{22}\dots u_{nn}$ equals the product of the pivots.

More generally, if $\mathbf{A}$ is nonsingular, and requires k row interchanges to arrive at its permuted factorization $\mathbf{PA} = \mathbf{LU}$, then $\det \mathbf{A} = \det \mathbf{P} \det \mathbf{U} = (-1)^k u_{11}u_{22}\dots u_{nn}$. Finally, $\mathbf{A}$ is singular **iff** $\det \mathbf{A} = 0$.

Proof: [regular/nonsingular] If $\mathbf{A}$ is regular, we need only the type-1 row operation (row replacement) to reduce $\mathbf{A}$ to upper triangular $\mathbf{U}$, and axiom-1 says these ($\mathbf{L}$) don't change the determinant.
Therefore, $|\mathbf{A}| = |\mathbf{LU}| = |\mathbf{U}|$, the formula for the latter being given by axiom-4.

The nonsingular case ($\mathbf{PA} = \mathbf{LU}$) follows in a similar fashion. $|\mathbf{PA}| = |\mathbf{LU}| \Rightarrow |\mathbf{PA}| = |\mathbf{U}|$.

By axiom-2, row interchanges (from $\mathbf{P}$) change the sign of the determinant, and so $|\mathbf{A}|$ equals $|\mathbf{U}|$ if there's been an even # of interchanges, but $-|\mathbf{A}|$ equals $|\mathbf{U}|$ if there's been an odd #.
So, $(-1)^k |\mathbf{A}| = |\mathbf{U}|$ or $|\mathbf{A}| = (-1)^k u_{11}u_{22}\dots u_{nn}$.

[singular] Finally, if $\mathbf{A}$ is singular, then we can reduce it to having at least one row of zeros by type 1 and/or type 2 row operations.

A previous lemma implies that the resulting matrix has 0 determinant, and so $|\mathbf{A}| = 0$, also.

Conversely, if $\mathbf{A}$ is nonsingular, all pivots are nonzero, so the signed product of pivots is nonzero.

Thus $|\mathbf{A}| = 0 \Rightarrow \mathbf{A}$ is singular. This proves both directions of the singularity criterion. ■

These calculations also prove uniqueness: any function satisfying the axioms must give the same signed product in the nonsingular case and zero in the singular case. Thus elimination forces the determinant of every square matrix.

Example: Compute $\det \begin{bmatrix} 5 & 6 & 6 & 6 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix}$.

We swap rows to obtain a convenient first pivot. This example also scales two rows to simplify the arithmetic, so we must record those factors as well as the sign change. Type-1 operations do not affect the determinant.

$$\xRightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 2 & 2 & 8 \\ 5 & 6 & 6 & 6 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \xRightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 5 & 6 & 6 & 6 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \xRightarrow{\text{type 1}} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & -4 & -16 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\xRightarrow{\text{type 1}} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 3 & 13 \end{bmatrix} \xRightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

$$\xRightarrow{\text{type 1}} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} =: \mathbf{U}.$$

- The final upper triangular matrix $\mathbf{U}$ has determinant 1.
- The swap contributes a factor of -1 , and
- the row scalings contribute $\frac{1}{2}$ and $-\frac{1}{4}$.

Hence $\mathbf{U}$'s determinant is: $\det \mathbf{U} = (-1)(\frac{1}{2})(-\frac{1}{4}) \det \mathbf{A} = \frac{1}{8} \det \mathbf{A}$.

Undoing these factors gives the determinant of the original matrix.

$$\text{Therefore: } \det \mathbf{A} = \det \begin{bmatrix} 5 & 6 & 6 & 6 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} = 8 \det \mathbf{U} = 8 \cdot 1 = 8.$$

Since the determinant is $8 \neq 0$, the original matrix is nonsingular.

The calculation illustrates why multiplying the final diagonal entries is only part of the work:
row swaps and row scalings must also be accounted for.

The following properties connect determinants with other matrix operations.
They can often save us from computing a determinant directly.

Proposition: The determinant of the product of two square matrices is the product of their determinants: $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

Proposition: If $\mathbf{A}$ is a nonsingular matrix, then $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$.

Proof: Since $\mathbf{A}$ is nonsingular, its inverse exists and $\det \mathbf{A} \neq 0$.

Thus we have: $\mathbf{AA}^{-1} = \mathbf{I}$.

Then, using the above multiplicative determinant property on this eqn we have:

Left-Hand Side: $|\mathbf{AA}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}|$. Right-Hand Side: $|\mathbf{I}| = 1$.

So, $|\mathbf{A}||\mathbf{A}^{-1}| = 1$. And dividing by (nonzero) $|\mathbf{A}|$ gives us:

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}. \quad \blacksquare$$

Proposition: Transposing a matrix does not change its determinant: $|\mathbf{A}^T| = |\mathbf{A}|$.

Consequences: Transposition turns rows into columns. Therefore,

- the same determinant rules hold for **column operations**, and
- a lower triangular matrix also has determinant equal to the product of its diagonal entries.

 Choose row or column operations according to which makes the calculation simpler!

Video Tutorials (visually rich and intuitive): <https://youtu.be/uQhTuRIWMxw>
<https://youtu.be/Ip3X9LOh2dk>