

Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

1.8 General Linear Systems

$$\begin{bmatrix} 0 & 7 & 3 & 4 & 1 \\ 0 & 0 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon: the zeros form a staircase

Definition: A matrix $\mathbf{U}$ is said to be in **echelon form** provided it has the following two properties (staircase properties):

- Every row of $\mathbf{U}$ that contains only zeros (if any) lies beneath every row that contains a nonzero element.
- In the other rows, the first nonzero element (**pivot**) lies strictly to the right of the first nonzero element in the preceding row (if there is a preceding row).

For $\mathbf{A}\vec{x} = \vec{b}$, after reduction to echelon form, the variables $\{x_i\}$ corresponding to the columns with pivots, are referred to as **basic variables**. The other variables are referred to as **free variables**.

$$\begin{bmatrix} 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So if our reduction resulted in the above echelon matrix, we would have 2 and 3 as pivots, with x_2, x_4 as basic variables and x_1, x_3 as free variables.

Proposition: Every matrix can be reduced to row echelon form by a sequence of type-1 and/or type-2 row operations.
[Gaussian reduction process ensures this is true]

Definition: The *rank* of a matrix is the number of pivots in its row echelon form.

Proposition: $\mathbf{A}^{n \times n}$ is nonsingular **iff** $\mathbf{A}$'s rank is equal to n . [Proof: see the definition of nonsingular]

Theorem: A system $\mathbf{A}\vec{x} = \vec{b}$ has either 0, 1, or ∞ solutions.

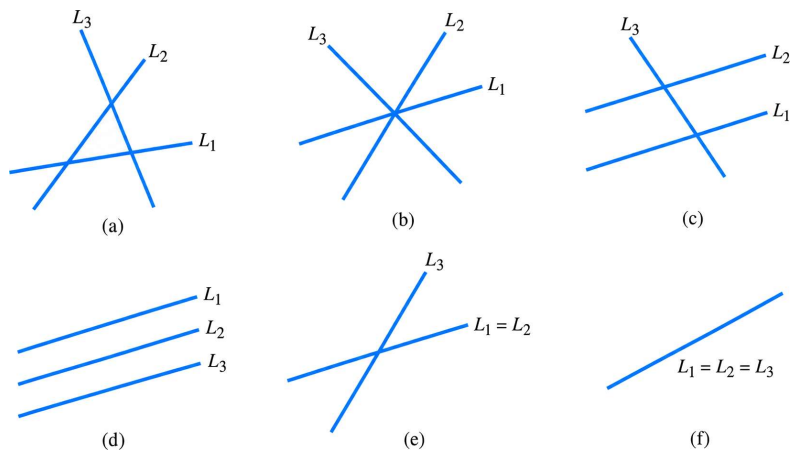
Example: $a_1x + b_1y = c_1$

The linear system: $a_2x + b_2y = c_2$

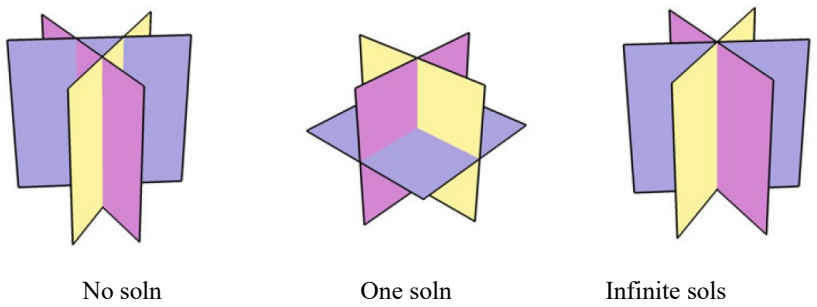
$$a_3x + b_3y = c_3$$

of three equations in two unknowns (x, y) represents three lines L_1 , L_2 , and L_3 in the xy -plane.

The figures below show possible configurations of these lines. Describe the soln set of each:



In higher dimension ($a_1x + b_1y + c_1z = d_1$, etc.):



Homogeneous Systems

Definition: A system $A\vec{x} = \vec{b}$ is *homogeneous* if $\vec{b}$ is a zero vector.
 Otherwise, it is *nonhomogeneous*.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 1 \\ a_{21} & a_{22} & a_{23} & | & 0 \end{bmatrix}$$

homogeneous *nonhomogeneous*

Definition: A soln to $A\vec{x} = \vec{b}$ is called **trivial** if the soln consists of $x_1 = x_2 = \dots = x_n = 0$.

You should convince yourself that the trivial soln is always a soln to any homogeneous system.

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 \\ a_{31} & a_{32} & a_{33} & | & 0 \\ a_{41} & a_{42} & a_{43} & | & 0 \end{bmatrix} \Rightarrow \begin{matrix} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \\ a_{41}x + a_{42}y + a_{43}z = 0 \end{matrix} \Rightarrow \begin{matrix} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 = 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 = 0 \\ a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 0 = 0 \\ a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 0 = 0 \end{matrix}$$

Observe this is true irrespective of what the a_{ij} are.

What if you knew that $\mathbf{H}$ had at least one *nontrivial* soln? Could you conclude something about how many sols it has?

What about a homogeneous system $\mathbf{H}$ with more variables (columns) than equations (rows)? How many sols?

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array} \right]$$

$$\rightarrow \quad \widetilde{\mathbf{H}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & 0 \\ 0 & 1 & c_{23} & 0 \end{array} \right] \quad ??$$

Note that we have no choice but to have a free variable. In this case, z .

Observe that it doesn't matter what the coefficients are, we will always have infinitely many sols.

For example, if $c_{13} = c_{23} = 0$, then z can be any real number a , and then we have the infinite sols $(0, 0, a)$.

Alternatively, if $c_{13} \neq 0$ and/or $c_{23} \neq 0$, observe that the eqns tell us that: $x = -c_{13}z$ and $y = -c_{23}z$,
and we can once again set z to be any real number a , and we have the infinite sols $(-c_{13}a, -c_{23}a, a)$, for all $a \in \mathbb{R}$.

Definition: A system of eqns is called *consistent* if it has at least one soln.

Theorem: A homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ of m eqns in n variables has a nontrivial soln $\vec{x} \neq \vec{0}$ iff the rank r of $\mathbf{A}$ is less than n .

If $m < n$, the system always has a nontrivial soln. If $m = n$ (square), the system has a nontrivial soln **iff** $\mathbf{A}$ is singular.

Proof [Rank]: Row reduction preserves the soln set, so we can evaluate the sols after reduction.

Denote r as the # of pivot variables, and thus we have $n - r$ free variables.

If $r < n$, there is at least one free variable. Set one free variable equal to 1 and all other free variables equal to 0, then solve for the pivot variables. The resulting soln is nontrivial because one of its entries is 1.

If $r = n$, there are no free variables; back substitution in the homogeneous system forces every variable to be 0. Thus a nontrivial soln exists exactly when $r < n$.

[Matrix Size] For the 2nd statement, since there is at most one pivot per row, $r \leq m$.

Hence $m < n$ implies $r < n$, so a nontrivial soln must exist (due to the earlier result).

So when $m = n$ (square), having $r = n$ is equivalent to $\mathbf{A}$ being nonsingular (trivial soln only).

Therefore, in the square case, a nontrivial soln exists exactly when $\mathbf{A}$ is singular. ■

Homogeneous systems are consistent because of the zero soln.