

Applied Linear Algebra

Textbook: Olver and Shakiban, Applied Linear Algebra, 2nd ed.

1.4 - 1.5 Pivoting and Permutations; Matrix Inverses

How do you solve $\mathbf{A}\vec{x} = \vec{b}$ if $\mathbf{A}$ isn't regular?

Example:
$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & b_1 \\ 1 & 3 & 4 & b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right]$$

Observe this is just the system: $x_2 + 2x_3 = b_1$, $x_1 + 3x_2 + 4x_3 = b_2$, and $x_3 = b_3$.

Obviously, the order in which we list these equations doesn't change the soln to the system.

So we're allowed to list them as: $x_1 + 3x_2 + 4x_3 = b_2$, $x_2 + 2x_3 = b_1$, and $x_3 = b_3$.

This gives us:
$$\left[\tilde{\mathbf{A}} | \vec{b} \right] = \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 0 & 1 & 2 & b_1 \\ 0 & 0 & 1 & b_3 \end{array} \right],$$

where $\tilde{\mathbf{A}}$ is now regular, and this system has the same sols as the original $\mathbf{A}$.

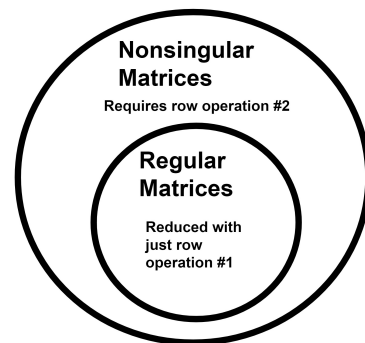
This justifies the row operation #2 of interchanging two rows, or *pivoting*.

Definition: A square matrix is called *non-singular* if it can be reduced to upper triangular form with all nonzero diagonal elements by applying row operations of types #1 and #2. (i.e., add scalar multiple of a row to a lower row, and/or pivots)

Theorem: $\mathbf{A}\vec{x} = \vec{b}$ has a unique soln for every choice of $\vec{b}$ **iff** $\mathbf{A}$ is square & nonsingular.

Proof of $\Leftarrow$: Nonsingularity implies reduction to upper triangular $\tilde{\mathbf{A}}$, having just one soln.

Proof of $\Rightarrow$: Section 1.8.



Observe that interchanging rows of a matrix can be accomplished by an elementary matrix, for example:

If $\mathbf{P} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$, this will interchange the first two rows of a 3×3 matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix} = \begin{bmatrix} b & b & b \\ a & a & a \\ c & c & c \end{bmatrix}.$$

Definition: A *permutation matrix* $\mathbf{P}$ is obtained from the identity matrix by applying any combination of row interchanges.

Lemma: $\mathbf{P}$ is a permutation matrix **iff** each row of $\mathbf{P}$ contains all 0 entries except for a single 1, and in addition, each column of $\mathbf{P}$ also contains all 0 entries except for a single 1.

Permuted LU Factorization

Note: For nonsingular matrices, to convert them to upper triangular form $\mathbf{U}$, it's nice if we have the fore-knowledge to perform the necessary pivots $\mathbf{P}$ first, and subsequently perform the required type 1 row operations $\mathbf{L}$. Because then we can recognize $\mathbf{PA}$ as regular, and by previous theorem it can be factored as $\mathbf{PA} = \mathbf{LU}$.

Practically speaking, though, we often use Gaussian elimination to reduce $\mathbf{A}$ to $\mathbf{U}$, where we often swap rows (pivot) after having already used a type-1 operation. Below you'll find an algorithm for calculating $\mathbf{P}, \mathbf{L}, \mathbf{U}$ in this general situation.

Permuted LU Factorization Algorithm

- Start w/two identity matrices: one will become $\mathbf{P}$, and one will become $\mathbf{L}$.
 - When you swap two rows in $\mathbf{A}$ (pivot): swap the same two rows in $\mathbf{P}$.
Then, in $\mathbf{L}$, swap only the entries you have already written down from earlier elimination steps. Leave the rest of $\mathbf{L}$ alone.
 - When you eliminate an entry in $\mathbf{A}$ (type-1 operation): don't perform that row operation on $\mathbf{L}$.
Instead, place -1 times the multiplier you used into the matching spot below the diagonal in $\mathbf{L}$.
- ❗ Keep the 1s on the diagonal of $\mathbf{L}$!

When you've generated your upper triangular $\mathbf{U}$, you should have $\mathbf{PA} = \mathbf{LU}$.

Example: Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -5 \\ 4 & -1 & -6 \\ 2 & -2 & 0 \end{bmatrix}$.

Find permutation, lower triangular, and upper triangular matrices $\mathbf{P}, \mathbf{L}$, and $\mathbf{U}$ such that $\mathbf{PA} = \mathbf{LU}$.

So we start out with $\mathbf{L}_0 = \mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We notice right away the first row of $\mathbf{A}$ will not work, so we interchange it with the 3rd row: $\mathbf{A}_1 = \begin{bmatrix} 2 & -2 & 0 \\ 4 & -1 & -6 \\ 0 & 1 & -5 \end{bmatrix}$,

Recording this in $\mathbf{P}$ gives us $\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. (note we don't need to edit $\mathbf{L}$ here)

Then proceeding with a type-1 row operation ($R_2 \rightarrow R_2 + (-2R_1)$), we have:

$$\mathbf{A}_2 = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & -6 \\ 0 & 1 & -5 \end{bmatrix}. \text{ Recording this in } \mathbf{L}, \text{ gives us } \mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we would like to swap the 2nd and 3rd row, giving us:

$$\mathbf{A}_3 = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -5 \\ 0 & 3 & -6 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \quad (\text{note we } \textit{DID} \text{ need to edit } \mathbf{L} \text{ here})$$

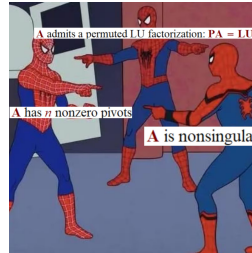
$$\text{Next, another type-1 operation: } \mathbf{A}_3 = \mathbf{U} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 9 \end{bmatrix}, \text{ recording this in } \mathbf{L}, \text{ gives us } \mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

$$\text{Therefore: } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -5 \\ 4 & -1 & -6 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 9 \end{bmatrix} \quad \text{OR} \quad \mathbf{P}_2 \mathbf{A} = \mathbf{L}_2 \mathbf{U}.$$

So, we managed to generalize LU factorization to matrices which require pivots.

Theorem: Given $\mathbf{A}^{n \times n}$. The following conditions are equivalent:

- ♦ $\mathbf{A}$ is nonsingular.
- ♦ $\mathbf{A}$ has n nonzero pivots.
- ♦ $\mathbf{A}$ admits a permuted LU factorization: $\mathbf{PA} = \mathbf{LU}$.



Matrix Inverses (§1.5)

Recall that for $a \in \mathbb{R}$, if $a \neq 0$, then there is a (unique) number $\frac{1}{a} = a^{-1} =: b$ such that $ab = ba = 1$.

We say b is a 's **inverse**, and we say both these nonzero numbers are **invertible**.

Does this exist for matrices? Perhaps $\mathbf{AB} = \mathbf{I}$? Yes, but we must change the assumption a bit.

Instead of $\mathbf{A} \neq 0$, we need something called the **determinant** of $\mathbf{A}$ to be nonzero.

Also, because matrices do not commute, we must check that both $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$.

Definition: We say $\mathbf{A}^{n \times n}$ is **invertible** if there exists $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

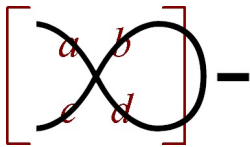
If $\mathbf{B}$ exists, then $\mathbf{B}$ is $\mathbf{A}$'s inverse, and is commonly denoted $\mathbf{A}^{-1}$.

In particular, $\mathbf{B}$ must be both a left- **and** a right-inverse.

! But only square matrices can multiply both ways, so only square matrices can be invertible.

2 × 2 Matrices

Lemma: Given: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\mathbf{A}$ is invertible **iff** $ad - bc$ (its **determinant**) is nonzero.



Proof: Let $\mathbf{X} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be its inverse. The right-inverse condition is:

$$\mathbf{AX} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving the system of four eqns: $x = \frac{d}{ad-bc}$, $y = -\frac{b}{ad-bc}$, $z = -\frac{c}{ad-bc}$, $w = \frac{a}{ad-bc}$, ...

provided $ad - bc \neq 0$.

In which case (in this 2×2 example) we have **inverse**: $\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.



One-over-Determinant, Interchange, then Negate (ODIN)

Inverse Properties

Theorem: $\mathbf{A}^{n \times n}$ has an inverse **iff** $\mathbf{A}$ is nonsingular. (proof provided later)

Lemma: The inverse $\mathbf{X}$ of a matrix $\mathbf{A}$ (if it exists) is unique. ...

Proof: Let $\mathbf{A}$ be invertible. Suppose $\mathbf{X}$ satisfies $\mathbf{XA} = \mathbf{AX} = \mathbf{I}$ and $\mathbf{Y}$ satisfies $\mathbf{YA} = \mathbf{AY} = \mathbf{I}$.

By associativity: $\mathbf{X} = \mathbf{XI} = \mathbf{X}(\mathbf{AY}) = (\mathbf{XA})\mathbf{Y} = \mathbf{IY} = \mathbf{Y}$. ■

Lemma: If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is also invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Proof: The matrix inverse equations $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{AA}^{-1}$ are sufficient to prove that $\mathbf{A}$ is the inverse of $\mathbf{A}^{-1}$. ■

Lemma: If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices of the same size, then their product $\mathbf{AB}$ is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Note that the order of the factors is reversed under inversion.

Proof: Let $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Then, by associativity, $\mathbf{X}(\mathbf{AB}) = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{AB} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$.

Also: $(\mathbf{AB})\mathbf{X} = \mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$.

Thus $\mathbf{X}$ is both a left and right inverse for the product matrix $\mathbf{AB}$. ■

Example: Show that if $\mathbf{A}$ is a nonsingular matrix, so is every power $\mathbf{A}^n$.

Recall that $\mathbf{A}$ is nonsingular **iff** it has an inverse $\mathbf{A}^{-1}$.

Now we can observe that $\mathbf{A}^n(\mathbf{A}^{-1})^n = (\mathbf{AA}\dots\mathbf{AA})(\mathbf{A}^{-1}\mathbf{A}^{-1}\dots\mathbf{A}^{-1}\mathbf{A}^{-1}) = (\mathbf{AA}\dots\mathbf{A})(\mathbf{AA}^{-1})(\mathbf{A}^{-1}\mathbf{A}^{-1}\dots\mathbf{A}^{-1})$

$$= (\mathbf{AA}\dots\mathbf{A})\mathbf{I}(\mathbf{A}^{-1}\mathbf{A}^{-1}\dots\mathbf{A}^{-1}) = (\mathbf{AA}\dots\mathbf{A})(\mathbf{A}^{-1}\mathbf{A}^{-1}\dots\mathbf{A}^{-1}) = \dots = \mathbf{AA}^{-1} = \mathbf{I}. \quad (\text{similarly, } (\mathbf{A}^{-1})^n\mathbf{A}^n = \mathbf{I}.)$$

Therefore, $\mathbf{A}^n$ has an inverse of $(\mathbf{A}^{-1})^n$, and therefore every power of $\mathbf{A}^n$ is nonsingular.

⚠ Warning: in general, $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

Gauss-Jordan Elimination

To find an inverse of a nonsingular square matrix, one generally uses the *Gauss-Jordan Elimination method*.

Justification

For square matrices $\mathbf{A}$, calculating the right-inverse turns out to also be the left-inverse,
so we need only calculate: $\mathbf{AX} = \mathbf{I}$, where we are solving for $\mathbf{X}$.

If we write $\mathbf{X} = [\vec{x}_1 \ \dots \ \vec{x}_n]$ (vectors are columns), then recall that $\mathbf{AX} = [\mathbf{A}\vec{x}_1 \ \dots \ \mathbf{A}\vec{x}_n]$.

So, solving $\mathbf{AX} = \mathbf{I}$ amounts to solving the eqns $\mathbf{A}\vec{x}_1 = \vec{e}_1, \dots, \mathbf{A}\vec{x}_n = \vec{e}_n$ or

$[\mathbf{A}|\vec{e}_1], \dots, [\mathbf{A}|\vec{e}_n]$, where the $\vec{e}_i$ are the standard unit vectors $(1, 0, 0, 0, \dots), (0, 1, 0, 0, \dots)$, etc.

Since each of these eqns has the same coefficient matrix $\mathbf{A}$, the n calculations will perform identical row operations.

This allows us to combine the calculations into a single calculation: $[\mathbf{A}|\vec{e}_1 \ \dots \ \vec{e}_n] = [\mathbf{A}|\mathbf{I}_n]$, or $\mathbf{AX} = \mathbf{I}$,
allowing us to make the changes to all the $\vec{e}_i$ simultaneously.

Our previous method asks us to reduce this to $[\mathbf{U}|\mathbf{C}]$, and then solve it using back substitution.

However, it is usually more convenient to continue using row operations until you have obtain $[\mathbf{I}|\mathbf{X}]$, or $\mathbf{IX} = \mathbf{X}$,

Accomplishing this may now require type-3 row operations (multiplying rows by nonzero constants).

So, convert $[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array} \right]$ into...

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a'_{11} & a'_{12} & \dots & a'_{1n} \\ 0 & 1 & 0 & 0 & a'_{21} & a'_{22} & \dots & a'_{2n} \\ 0 & 0 & 1 & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{array} \right] = [\mathbf{I} \mid \mathbf{A}^{-1}], \text{ using elementary row operations.}$$

Example: Use Gauss-Jordan elimination to find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R2+(-3R1)} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R3+(-R1)} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 6 & -5 & -3 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R3+(-6R1)} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 3 & 1 & -6 \end{array} \right] =: [\mathbf{U} \mid \tilde{\mathbf{I}}].$$

At this point, we have reduced the original system $\mathbf{AX} = \mathbf{I}$ to 3 equations $\mathbf{UX} = \tilde{\mathbf{e}}_i$

But let's continue to $[\mathbf{I} \mid \mathbf{A}^{-1}]$.

$$\xrightarrow{R1+2R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 3 & 1 & -6 \end{array} \right] \xrightarrow{-\frac{1}{5}R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array} \right]$$

(notice how I am allowing myself to add constant multiples of lower rows to upper rows!)

(also notice how I used type-3 operation, and avoided fractions until the last possible moment)

$$\mathbf{A}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 & -2 \\ -5 & 0 & 5 \\ -3 & -1 & 6 \end{bmatrix}.$$



Always Be Adding
AVOID FRACTIONS!!!



Elementary Matrices: Type 3

Elementary matrix which performs scalar multiplication of i^{th} row by c .

$$\mathbf{E} = [\vec{e}_1 \dots c\vec{e}_i \dots \vec{e}_n]. \text{ For example: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{With inverse } \mathbf{E}^{-1} = [\vec{e}_1 \dots \frac{1}{c}\vec{e}_i \dots \vec{e}_n] \text{ For example: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Verify this is the inverse!}$$

$$\text{So, if we want to multiply the third row of a } 3 \times 3 \text{ matrix by } -2, \text{ we can do so with: } \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Lemma: Every elementary matrix is nonsingular, and its inverse is also an elementary matrix of the same type.

Proof: Trivial (try to convince yourself!)

We now have sufficient results to prove the following (previously mentioned) theorem in the $\Leftarrow$ direction:

Theorem: $\mathbf{A}^{n \times n}$ has an inverse **iff** $\mathbf{A}$ is nonsingular.

Proof ($\Leftarrow$): Assuming $\mathbf{A}$ is nonsingular, we must show that $\mathbf{A}^{n \times n}$ has an inverse.

Gauss-Jordan method reduces nonsingular $\mathbf{A}^{n \times n}$ to $\mathbf{I}_n$ through row operations.

Let $\mathbf{E}_1, \dots, \mathbf{E}_N$ be the corresponding elementary matrices. So: $\mathbf{E}_N \mathbf{E}_{N-1} \dots \mathbf{E}_1 \mathbf{A} = \mathbf{I}_n$.

I claim $\mathbf{X} = \mathbf{E}_N \mathbf{E}_{N-1} \dots \mathbf{E}_1$ is the inverse of $\mathbf{A}$.

We already see that it is the left inverse. Furthermore, each elementary matrix has an inverse.

Therefore, $\mathbf{X}$ is itself invertible and: $\mathbf{X}^{-1} = (\mathbf{E}_N \mathbf{E}_{N-1} \dots \mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \dots \mathbf{E}_N^{-1}. \quad (*)$

So, multiplying $\mathbf{X}\mathbf{A} = \mathbf{I}$ on the left by $\mathbf{X}^{-1}$ leads to $\mathbf{A} = \mathbf{X}^{-1}$.

And by previous Lemma, the inverse $\mathbf{X}^{-1}$ of an invertible matrix $\mathbf{X}$ is invertible. Thus, $\mathbf{A}$ is invertible. ■

Furthermore, if we substitute $\mathbf{A} = \mathbf{X}^{-1}$ into $(*)$, we get the following proposition:

Proposition: Every nonsingular matrix can be written as the product of elementary matrices.

Example: Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$, if possible, by applying the Gauss-Jordan Method.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2+(-2R_1)} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array} \right] \xrightarrow{R_1+3R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array} \right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{(-1) \cdot R_2 \text{ \& } (-1) \cdot R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array} \right].$$

Therefore, $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$

Solving Linear Systems with the Inverse

Theorem: If $\mathbf{A}$ is nonsingular, then the unique soln to $\mathbf{A}\vec{x} = \vec{b}$ is: $\vec{x} = \mathbf{A}^{-1}\vec{b}$.

Proof: We merely multiply the system (on the left) by $\mathbf{A}^{-1}$, which yields $\vec{x} = \mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{b}$.

Verifying this is a soln, note that: $\mathbf{A}\vec{x} = \mathbf{A}(\mathbf{A}^{-1}\vec{b}) = \vec{b}$. ■

Example: Solve the following system of linear equations by computing the inverse of its coefficient matrix.

$3u - v = 2$ and $u + 5v = 12$.

$$\text{Let: } \mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}. \quad \Rightarrow \mathbf{A}\vec{x} = \vec{b}, \text{ where } \vec{x} := (u, v) \text{ and } \vec{b} := (2, 12).$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \mathbf{A}^{-1}\vec{b} = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{11}{8} \\ \frac{17}{8} \end{bmatrix}.$$

So, $u = \frac{11}{8}$ and $v = \frac{17}{8}$ is the unique soln. ■

The LDV Factorization

Theorem (regular $\Leftrightarrow$ LDV): $\mathbf{A}$ is regular **iff** it admits a factorization $\mathbf{A} = \mathbf{LDV}$, where $\mathbf{L}$ is lower unitriangular, $\mathbf{D}$ is diagonal with nonzero diagonal entries, and $\mathbf{V}$ is an upper unitriangular.

In particular, once one has calculated $\mathbf{A} = \mathbf{LU}$, then $\mathbf{D}$ is a diagonal matrix consisting of the same diagonal entries as $\mathbf{U}$, that is, the pivots. $\mathbf{V}$ is then obtained from $\mathbf{U}$ by dividing each row by its pivot.

- **LU tells us:** elimination produces a lower-triangular part and an upper-triangular part.
- **LDV asks:** can we separate the actual scaling from the row-addition information?

This decomposition method shows up in the applied areas of:

- Optimization,
- Fitting data to a curve, and
- Physical systems (e.g. ones that behave like springs).

In these contexts, symmetric matrices appear, and this type of decomposition becomes crucial.

Proposition (decomposition uniqueness): If $\mathbf{A} = \mathbf{LU}$ is regular, then the factors $\mathbf{L}$ and $\mathbf{U}$ are uniquely determined. The same holds for the $\mathbf{A} = \mathbf{LDV}$ factorization. [Proof in textbook]

Theorem (nonsingular $\Leftrightarrow$ permuted LDV): $\mathbf{A}$ is nonsingular **iff** there is a permutation matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{LDV}$, where $\mathbf{L}$ is a lower unitriangular matrix, $\mathbf{D}$ is a diagonal matrix with nonzero diagonal entries, and $\mathbf{V}$ is an upper unitriangular matrix.

Proof: If $\mathbf{A}$ is regular, then the regular LDV factorization theorem above is sufficient (where $\mathbf{P} = \mathbf{I}$).

If $\mathbf{A}$ is not regular, but nonsingular, observe we noted earlier that $\mathbf{A}$ is nonsingular **iff** there is a

permutation matrix $\mathbf{P}$ such that $\mathbf{PA}$ is regular. Applying the regular LDV factorization theorem on $\mathbf{PA}$ completes this proof. ■

Example: Produce the **LDV** or a permuted **LDV** factorization of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{bmatrix}$.

Recall from a previous example (see previous lecture) that we had generated the **LU** factorization for **A** as:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{LU}.$$

Generating the diagonal **D** from **U**'s pivots: $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Then dividing each of **U**'s rows by their pivots, we get: $\mathbf{V} = \begin{bmatrix} \frac{1}{2} \cdot \vec{r}_1 \\ \frac{1}{3} \cdot \vec{r}_2 \\ -1 \cdot \vec{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{bmatrix} = \mathbf{LDV}.$