

Worksheet 8 — QR, compatibility, and constrained geometry

Sections 4.3–4.4 and 5.2 • Student worksheet

Learning goals. Connect orthogonality to matrix products, construct a QR factorization, use the left nullspace as a consistency test, and minimize a quadratic geometrically.

Name(s): _____

Section/group: _____

Core practice

1. Products of orthogonal matrices

Prove that if Q_1 and Q_2 are square orthogonal matrices of the same size, then $Q_1 Q_2$ is orthogonal. State where order reversal under transpose is used.

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2. Build and check a QR factorization

Find a QR factorization of A by applying Gram–Schmidt to its columns. Make the diagonal entries of R positive and verify one off-diagonal entry using a dot product.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

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3. Compatibility from the left nullspace

Find a nonzero vector in $\ker(A^T)$. Use it to state a necessary and sufficient condition on $b = (b_1, b_2, b_3)^T$ for $Ax = b$ to be solvable.

$$A = \begin{bmatrix} 3 & -3 \\ -2 & 0 \\ -1 & -4 \end{bmatrix}.$$

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4. A quadratic bowl and its lowest point

Let $q(x) = x^T Kx - 2f^T x + 2$, where

$$K = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Show that K is positive definite, find the lowest point of the graph $z = q(x)$, and give its height.

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