

Worksheet 5 — Fundamental subspaces and graph matrices

Sections 2.5–2.6 • Student worksheet

Learning goals. Read bases and dimensions from echelon data, connect row and column spaces, and interpret the kernels of an incidence matrix.

Name(s): _____

Section/group: _____

Core practice

1. Four fundamental subspaces from one echelon form

The matrix A and a row-echelon form U are shown below. Find bases for $\text{im}(A)$, $\ker(A)$, $\text{im}(A^T)$, and $\ker(A^T)$. State the dimension of each space.

$$A = \begin{bmatrix} 1 & 4 & -5 & 0 & 2 \\ 2 & 8 & 0 & 1 & 7 \\ -1 & -4 & 15 & 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 & -5 & 0 & 2 \\ 0 & 0 & 10 & 1 & 3 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Remember: pivot columns for $\text{im}(A)$ must be taken from the original matrix A , not from U . Row operations do preserve the row space.

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2. Dimension bookkeeping

Let C be a 3×4 matrix of rank 2. Without knowing its entries, determine the dimensions of $\ker(C)$, $\text{im}(C)$, $\text{im}(C^T)$, and $\ker(C^T)$. Identify the ambient space for each.

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3. Cycle space and component space of a directed graph

A directed graph has vertices 1,2,3,4 and edges e_1 : $1 \rightarrow 2$, e_2 : $2 \rightarrow 3$, e_3 : $3 \rightarrow 1$, e_4 : $3 \rightarrow 4$. Its incidence matrix uses -1 at the tail and $+1$ at the head.

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find bases for $\ker(B)$ and $\ker(B^T)$. Interpret each basis vector using the graph. What is $\text{rank}(B)$?

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Extension / challenge

4. Extension: consistency from the cokernel

Use your description of $\ker(B^T)$ to state the exact condition on $b \in \mathbb{R}^4$ for $Bx = b$ to be solvable. Explain the condition as conservation of flow.

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