Celestial Mechanics

These notes were gathered from many sources to prepare for an oral exam.

History

Modern analytic celestial mechanics started with Isaac Newton's Principia of 1687. The name "celestial mechanics" is more recent than that. Newton wrote that the field should be called "rational mechanics." The term "dynamics" came in a little later with Gottfried Leibniz, and over a century after Newton, Pierre-Simon Laplace introduced the term "celestial mechanics." Prior to Kepler there was little connection between exact, quantitative prediction of planetary positions, using geometrical or arithmetical techniques, and contemporary discussions of the physical causes of the planets' motion.

Johannes Kepler

Johannes Kepler (1571–1630) was the first to closely integrate the predictive geometrical astronomy, which had been dominant from Ptolemy in the 2nd century to Copernicus, with physical concepts to produce a New Astronomy, Based upon Causes, or Celestial Physics in 1609. His work led to the modern laws of planetary orbits, which he developed using his physical principles and the planetary observations made by Tycho Brahe. Kepler's model greatly improved the accuracy of predictions of planetary motion, years before Isaac Newton developed his law of gravitation in 1686.

Isaac Newton

Isaac Newton (1642–1727) is credited with introducing the idea that the motion of objects in the heavens, such as planets, the Sun, and the Moon, and the motion of objects on the ground, like cannon balls and falling apples, could be described by the same set of physical laws. In this sense he unified celestial and terrestrial dynamics. Using Newton's law of universal gravitation, proving Kepler's Laws for the case of a circular orbit is simple. Elliptical orbits involve more complex calculations, which Newton included in his Principia.

Joseph-Louis Lagrange

After Newton, Lagrange (1736–1813) attempted to solve the three-body problem, analyzed the stability of planetary orbits, and discovered the existence of the Lagrangian points. Lagrange also reformulated the principles of classical mechanics, emphasizing energy more than force and developing a method to use a single polar coordinate equation to describe any orbit, even those that are parabolic and hyperbolic. This is useful for calculating the behaviour of planets and comets and such. More recently, it has also become useful to calculate spacecraft trajectories.

Simon Newcomb

Simon Newcomb (1835–1909) was a Canadian-American astronomer who revised Peter Andreas Hansen's table of lunar positions. In 1877, assisted by George William Hill, he recalculated all the major astronomical constants. After 1884, he conceived with A. M. W. Downing a plan to resolve much international confusion on the subject. By the time he attended a standardisation conference in Paris, France in May 1886, the international consensus was that all ephemerides should be based on Newcomb's calculations. A further conference as late as 1950 confirmed Newcomb's constants as the international standard.

Albert Einstein

Albert Einstein (1879–1955) explained the anomalous precession of Mercury's perihelion in his 1916 paper The Foundation of the General Theory of Relativity. This led astronomers to recognize that Newtonian mechanics did not provide the highest accuracy. Binary pulsars have been observed, the first in 1974, whose orbits not only require the use of General Relativity for their explanation, but whose evolution proves the existence of gravitational radiation, a discovery that led to the 1993 Nobel Physics Prize.

Classical Mechanics

Describes the motion of macroscopic objects, from projectiles to parts of machinery, and astronomical objects, such as spacecraft, planets, stars and galaxies. If the present state of an object is known it is possible to predict by the laws of classical mechanics how it will move in the future (determinism) and how it has moved in the past (reversibility).

The earliest development of classical mechanics is often referred to as Newtonian mechanics. It consists of the physical concepts employed by and the mathematical methods invented by Isaac Newton ($F = ma$) and Gottfried Wilhelm Leibniz and others in the 17th century to describe the motion of bodies under the influence of a system of forces.

Classical mechanics provides extremely accurate results when studying large objects that are not extremely massive (so, not black holes) and speeds not approaching the speed of light. When the objects being examined have about the size of an atom diameter, it becomes necessary to introduce the other major sub-field of mechanics: **quantum mechanics**. To describe velocities that are not small compared to the speed of light, **special relativity** is needed. In case that objects become extremely massive, **general relativity** becomes applicable. However, a number of modern sources do include relativistic mechanics into classical physics, which in their view represents classical mechanics in its most developed and accurate form.

Lagrangian Mechanics

A reformulation of classical mechanics, introduced by the Italian-French mathematician and astronomer Joseph-Louis Lagrange in 1788.

In Lagrangian mechanics, the trajectory of a system of particles is derived by solving the Lagrange equations in one of two forms: either the **Lagrange equations of the first kind**, which treat constraints explicitly as extra equations, often using Lagrange multipliers; or the **Lagrange equations of the second kind**, which incorporate the constraints directly by judicious choice of generalized coordinates. In each case, a mathematical function called the Lagrangian is a function of the generalized coordinates, their time derivatives, and time, and contains the information about the dynamics of the system.

No new physics are necessarily introduced in applying Lagrangian mechanics compared to Newtonian mechanics. It is, however, more mathematically sophisticated and systematic. Newton's laws can include non-conservative forces like friction; however, they must include constraint forces explicitly and are best suited to Cartesian coordinates. Lagrangian mechanics is ideal for systems with conservative forces and for bypassing constraint forces in any coordinate system. Dissipative and driven forces can be accounted for by splitting the external forces into a sum of potential and non-potential forces, leading to a set of **modified Euler–Lagrange** (**EL**) **equations**. Generalized coordinates can be chosen by convenience, to exploit symmetries in the system or the geometry of the constraints, which may simplify solving for the motion of the system. Lagrangian mechanics also reveals conserved quantities and their symmetries in a direct way, as a special case of Noether's theorem.

Intuitive Derivation of the Euler Lagrange Equations

https://farside.ph.utexas.edu/teaching/336L/Fluid/node266.html.

Hamiltonian Mechanics:

Hamiltonian mechanics is a theory developed as a reformulation of classical mechanics and predicts the same outcomes as non-Hamiltonian classical mechanics. It uses a different mathematical formalism, providing a more abstract understanding of the theory. Historically, it was an important reformulation of classical mechanics, which later contributed to the formulation of statistical mechanics and quantum mechanics. Hamiltonian mechanics was first formulated by William Rowan Hamilton in 1833, starting from Lagrangian mechanics. In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates $r = (q,p)$, where each component of the coordinate q_i, p_i is indexed to the frame of reference of the system.

The time evolution of the system is uniquely defined by Hamilton's equations: $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$, $\frac{dq}{dt} = \frac{\partial H}{\partial p}$, where $H = H(q,p,t)$ is the Hamiltonian, which often corresponds to the total energy of the system. For a closed system, it is the sum of the kinetic and potential energy in the system.

In Newtonian mechanics, the time evolution is obtained by computing the total force being exerted on each particle of the system, and from Newton's second law, the time-evolutions of both position and velocity are computed. In contrast, in Hamiltonian mechanics, the time evolution is obtained by computing the Hamiltonian of the system in the generalized coordinates and inserting it in the Hamilton's equations. This approach is equivalent to the one used in Lagrangian mechanics. In fact, the Hamiltonian is the Legendre transform of the Lagrangian when holding q and t fixed and defining p as the dual variable, and thus both approaches give the same equations for the same generalized momentum. The main motivation to use Hamiltonian mechanics instead of Lagrangian mechanics comes from the symplectic structure of Hamiltonian systems.

While Hamiltonian mechanics can be used to describe simple systems such as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time, its strength is shown in more complex dynamic systems, such as planetary orbits in celestial mechanics. The more degrees of freedom the system has, the more complicated its time evolution is and, in most cases, it becomes chaotic.

Hamiltonian System:

A Hamiltonian system is a dynamical system governed by Hamilton's equations. In physics, this dynamical system describes the evolution of a physical system such as a planetary system or an electron in an electromagnetic field. These systems can be studied in both Hamiltonian mechanics and dynamical systems theory.

Informally, a Hamiltonian system is a mathematical formalism developed by Hamilton to describe the evolution equations of a physical system. The advantage of this description is that it gives important insight about the dynamics, even if the initial value problem cannot be solved analytically. One example is the planetary movement of three bodies: even if there is no simple solution to the general problem, Poincaré showed for the first time that it exhibits deterministic chaos.

Formally, a Hamiltonian system is a dynamical system completely described by the scalar function $H(q,p,t)$, the Hamiltonian. The state of the system, r, is described by the generalized coordinates "momentum" p and "position" q where both p and q are vectors with the same dimension N. So, the system is completely described by the $2N$ dimensional vector $r = (q,p)$, and the evolution equation is given by the Hamilton's equations: $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$ and $\frac{dq}{dt} = \frac{\partial H}{\partial p}$.

The trajectory $r(t)$ is the solution of the initial value problem defined by the Hamilton's equations and the initial condition $r(0) = r_0 \in \mathbb{R}^{2N}$.

Integral of Motion:

A **constant of motion** may be defined in a given force field as any function of phase-space coordinates (position and velocity, or position and momentum) **and time** that is constant throughout a trajectory. A subset of the constants of motion are the **integrals of motion**, or **first integrals**, defined as any functions of only the phase-space coordinates that are constant along an orbit. Every integral of motion is a constant of motion, but the converse is not true because a constant of motion may depend on time. Examples of integrals of motion are the angular momentum vector, $L = r \times p$, or a Hamiltonian without time dependence, such as $H(x,v) = \frac{1}{2}mv^2 + \Phi(x)$. An example of a function that is a constant of motion but not an integral of motion would be the function $C(x,v,t) = x - vt$ for an object moving at a constant speed in one dimension.

Symplectic Matrix:

A symplectic matrix is a $2n \times 2n$ matrix M with real entries that satisfies the condition: $M^T \Omega M = \Omega$, where Ω is a fixed $2n \times 2n$ nonsingular, skew-symmetric matrix. This definition can be extended to $2n \times 2n$ matrices with entries in other fields, such as the complex numbers.

Typically Ω is chosen to be the "Poission" symplectic block matrix: $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. The matrix Ω has determinant +1 and has an inverse given by $\Omega - 1 = \Omega^T = -\Omega$.

Every symplectic matrix has determinant 1, and the $2n \times 2n$ symplectic matrices with real entries form a subgroup $Sp(2n, \mathbb{R})$ of the special linear group $SL(2n, \mathbb{R})$ (set of $n \times n$ matrices with determinant 1). Topologically, the symplectic group $Sp(2n, \mathbb{R})$ is a connected noncompact real Lie group of real dimension $n(2n + 1)$. The symplectic group can be defined as the set of linear transformations that preserve the symplectic form of a real symplectic vector space. Examples of symplectic matrices include the identity matrix, the Poission matrix, and the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The Kepler Problem (Existence and Uniqueness):

In classical mechanics, the Kepler problem (formulated **after** Kepler) is a special case of the two-body problem. It starts by assuming two bodies interact by a central force F emanating from the central body, that varies in strength as the inverse square of distance r between them (via Newton). The force may be attractive or repulsive. The "problem" to be solved is to find the position or speed of the two bodies over time given their masses and initial positions and velocities. Using classical mechanics, the solution can be expressed as a Kepler orbit using six orbital elements.

The Kepler problem is named after Johannes Kepler, who proposed **Kepler's laws of planetary motion** (which are part of classical mechanics and solve the problem for the orbits of the planets) and investigated the types of forces that would result in orbits obeying those laws (called **Kepler's inverse problem**).

The Kepler problem in general relativity produces more accurate predictions, especially in strong gravitational fields.

The Kepler problem arises in many contexts, some beyond the physics studied by Kepler himself. The Kepler problem is important in celestial mechanics, since Newtonian gravity obeys an inverse square law. Examples include a satellite moving about a planet, a planet about its sun, or two binary stars about each other. The Kepler problem is also important in the motion of two charged particles, since Coulomb's law of electrostatics also obeys an inverse square law. Examples include the hydrogen atom, positronium and muonium, which have all played important roles as model systems for testing physical theories and measuring constants of nature.

The **Kepler problem** and the **simple harmonic oscillator problem** are the two most fundamental problems in classical mechanics. They are the only two problems that have closed orbits for every possible set of initial conditions, i.e., return to their starting point with the same velocity (**Bertrand's theorem**). The Kepler problem is often used to develop new methods in classical mechanics, such as **Lagrangian mechanics**, **Hamiltonian mechanics**, the **Hamilton–Jacobi equation**, and **action-angle coordinates**. The Kepler problem also conserves the **Laplace–Runge–Lenz vector**, which has since been generalized to include other interactions. The solution of the Kepler problem allowed scientists to show planetary motion could be explained entirely by classical mechanics and Newton's law of gravity; the scientific explanation of planetary motion played an important role in ushering in the Enlightenment.

Mathematical definition: The central force $\vec{F}$ that varies in strength as the inverse square of the distance r between them: $\vec{F} = \frac{k}{r^2}\hat{r}$, where k is a constant and $\hat{r}$ represents the unit vector along the line between them. The force may be either attractive ($k < 0$) or repulsive ($k > 0$). The corresponding scalar potential (the potential energy of the

non-central body) is: $V(r) = \frac{k}{r}$.

Kepler Problem: $\begin{cases} \dot{q} = p \\ \dot{p} = k\frac{q}{|q|^3} \end{cases}$. Let $\Delta = \{q : q_i = q_j, \text{ where } i \neq j\}$. Solutions exist for the Kepler problem on $\mathbb{R}^2 \backslash \Delta$. That is, away from all collisions. However, you can regularize collisions of 2 bodies. Then, solutions exist for all time.

Bertrand's Theorem

In classical mechanics, Bertrand's theorem states that among central-force potentials with bounded orbits, there are only two types such that all bounded orbits are also closed orbits: (1) an inverse-square central force such as the gravitational or electrostatic potential: $V(r) = -\frac{k}{r}$, and (2) the radial harmonic oscillator potential: $V(r) = \frac{1}{2}kr^2$.

Many conservative central force problems may have isolated circular orbits or other closed (i.e., periodic) orbits. But there are only two conservative central potentials for which all bounded orbits are necessarily closed. For both of these potentials, the closed orbits are all ellipses, only in the inverse square case, the force is directed toward one focus of the ellipse, while in the harmonic oscillator case, the force is directed toward the geometric center of the ellipse.

Condition for closed orbits is given by $\frac{d^2V_{eff}}{dr^2} > 0$. (Bertrand's theorem).

The equation of motion for the radius r of a particle of mass m moving in a central potential $V(r)$ is given by motion equations: $m\ddot{r} - mr\omega^2 = m\ddot{r} - \frac{L^2}{mr^3} = -\frac{dV}{dr}$, where $\omega \equiv \dot{\theta}$, and the angular momentum $L = mr^2\omega$ is conserved. For illustration, the first term on the left is zero for circular orbits, and the applied inwards force $\frac{dV}{dr}$ equals the centripetal force requirement $mr\omega^2$, as expected.

The definition of angular momentum allows a change of independent variable from t to θ. Observe $\frac{d\theta}{dt} = \frac{L}{mr^2}$, so multiplying by $\frac{d}{d\theta}$ gives $\frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\theta}$, giving the new equation of motion that is independent of time: $\frac{L}{r^2}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = -\frac{dV}{dr}$.

This equation becomes quasilinear on making the change of variables $u \equiv \frac{1}{r}$ and multiplying both sides by $\frac{mr^2}{L^2}$:

Observe that: $\frac{d}{dr} = \frac{d}{du}\frac{du}{dr} = -\frac{1}{r^2}\frac{d}{du} = -u^2\frac{d}{du} \quad \Rightarrow \quad \frac{L^2}{m}u^2\frac{d}{d\theta}\left(u^2\frac{d(\frac{1}{u})}{d\theta}\right) - \frac{L^2}{m}u^3 = u^2\frac{d}{du}V(\frac{1}{u})$

$\Rightarrow \quad -\frac{d}{d\theta}\left(-\frac{du}{d\theta}\right) + u = -\frac{m}{L^2}\frac{d}{du}V(\frac{1}{u}) \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{d}{du}V(\frac{1}{u})$.

As noted above, all **central** forces can produce circular orbits given an appropriate initial velocity. However, if some radial velocity is introduced (so orbit not circular), these orbits need not be stable (i.e., remain in orbit indefinitely) nor closed (repeatedly returning to exactly the same path). Here we show that stable, exactly closed orbits can be produced only with an inverse-square force or radial harmonic oscillator potential (a necessary condition). In the following sections, we show that those force laws do produce stable, exactly closed orbits (a sufficient condition).

Define $J(u)$ as : $\frac{d^2u}{d\theta^2} + u = J(u) \equiv -\frac{m}{L^2}\frac{d}{du}V(\frac{1}{u}) = -\frac{m}{L^2}\left(-\frac{dr}{du}\frac{d}{dr}\int_0^r V(r)ds\right) = -\frac{m}{L^2}\frac{dr}{du}f(r) = -\frac{m}{L^2u^2}f(\frac{1}{u}), \qquad (0)$

where f represents the radial force. The criterion for perfectly circular motion at a radius r_0 is that the first term on the left (u'') is zero: $u_0 = J(u_0) = -\frac{m}{L^2u_0^2}f(\frac{1}{u_0}), \qquad (1)$

where $u_0 \equiv \frac{1}{r_0}$.

The next step is to consider the equation for u under small perturbations $\eta \equiv u - u_0$ from perfectly circular orbits. The J function can be expanded in a standard Taylor series: $J(\eta) \equiv J(u_0) + J'(u_0)\eta + \frac{1}{2}J''(u_0)\eta^2 + \frac{1}{6}J'''(u_0)\eta^3 + \ldots$

Substituting this expansion into the equation (0) for u and subtracting the constant terms yields:

$\frac{d^2\eta}{d\theta^2} + \eta = \eta J'(u_0) + \frac{1}{2}\eta^2 J''(u_0) + \frac{1}{6}\eta^3 J'''(u_0) + \ldots,$

Which can be written as: $\frac{d^2\eta}{d\theta^2} + \beta^2\eta = \frac{1}{2}\eta^2 J''(u_0) + \frac{1}{6}\eta^3 J'''(u_0) + \ldots, \qquad (2)$

where $\beta^2 \equiv 1 - J'(u_0)$ is a constant. β^2 must be nonnegative; otherwise, the radius of the orbit would vary exponentially away from its initial radius. ($\beta = 0$ corresponds to a perfectly circular orbit.) If the right side may be neglected (i.e., for small perturbations), the solutions are $\eta(\theta) = h_1 \cos(\beta\theta)$, where the amplitude h_1 is a constant of integration. For the orbits to be closed, β must be a rational number. What's more, it must be the same rational number for all radii, since β cannot change continuously; the rational numbers are totally disconnected from one another. Using the definition of J along with equation (1),

$$J'(u_0) = \frac{2}{u_0}\left[\frac{m}{L^2 u_0^2} f\left(\frac{1}{u_0}\right)\right] - \left[\frac{m}{L^2 u_0^2} f\left(\frac{1}{u_0}\right)\right] \frac{1}{f\left(\frac{1}{u_0}\right)} \frac{df}{du} = -2 + \frac{u_0}{f\left(\frac{1}{u_0}\right)} \frac{df}{du} = 1 - \beta^2, \qquad (2.1)$$

where $\frac{df}{du}$ is evaluated at $\frac{1}{u_0}$. Since β is constant, this must hold for any value of u_0 (radius), we can let u_0 vary as u in (2.1). Observe that: $\frac{df}{du} = \frac{df}{dr}\frac{dr}{du} = -\frac{df}{dr}\frac{1}{u^2} = -\frac{df}{dr}\frac{r}{u}$. So, the rightmost equation in (2.1) can be written as $\frac{df}{dr} = (\beta^2 - 3)\frac{f}{r}$, which implies that the force must follow a power law: $f(r) = -\frac{k}{r^{3-\beta^2}}$. Hence, J must have the general form: $J(u) = \frac{mk}{L^2} u^{1-\beta^2}. \qquad (2.5)$

For more general deviations from circularity (i.e., when we cannot neglect the higher-order terms in the Taylor expansion of J), η may be expanded in a Fourier series, e.g.,

$$\eta(\theta) = h_0 + h_1 \cos\beta\theta + h_2 \cos 2\beta\theta + h_3 \cos 3\beta\theta + \ldots$$

We substitute this into equation (2) and equate the coefficients belonging to the same frequency, keeping only the lowest-order terms. As we show below, h_0 and h_2 are smaller than h_1, being of order h_1^2. h_3, and all further coefficients, are at least of order h_1^3. This makes sense, since $h_0, h_2, h_3, \ldots$ must all vanish faster than h_1 as a circular orbit is approached.

$$h_0 = h_1^2 \frac{J''(u_0)}{4\beta^2}, \qquad h_2 = -h_1^2 \frac{J''(u_0)}{12\beta^2}, \qquad h_3 = -\frac{1}{8\beta^2}\left[h_1 h_2 \frac{J''(u_0)}{2} + h_1^3 \frac{J'''(u_0)}{24}\right].$$

From the $\cos(\beta\theta)$ term, we get: $0 = (2h_1 h_0 + h_1 h_2)\frac{J''(u_0)}{2} + h_1^3 \frac{J'''(u_0)}{8} = \frac{h_1^3}{24\beta^2}\left(3\beta^2 J'''(u_0) + 5J''(u_0)^2\right), \qquad (3)$

where in the last step we substituted in the values of h_0 and h_2.

Using equations (3) and (1), we can calculate the second and third derivatives of J evaluated at u_0:
$J''(u_0) = -\frac{\beta^2(1-\beta^2)}{u_0}$, $J'''(u_0) = \frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}$.

So, $h_3 = -\frac{h_1^3}{192\beta^2}\left[J'''(u_0) - \frac{(J''(u_0))^2}{\beta^2}\right] = -\frac{h_1^3}{192\beta^2}\left[\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2} - \frac{\beta^4(1-\beta^2)^2}{\beta^2 u_0^2}\right]$

$$= -\frac{h_1^3}{96u_0^2}(1-\beta^2)\beta^2.$$

Substituting these values into the last equation (3) yields the main result of Bertrand's theorem:

$$0 = \frac{h_1^3}{24\beta^2}\left(3\beta^2 \frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2} + 5\frac{\beta^4(1-\beta^2)^2}{u_0^2}\right) \Rightarrow 0 = \frac{h_1^3}{24u_0^2}\beta^2(1-\beta^2)(3(1+\beta^2) + 5(1-\beta^2))$$

$$\Rightarrow \quad \beta^2(1-\beta^2)(4-\beta^2) = 0.$$

Observe that when $\beta = 1$, we have $J(0) \overset{(0)}{=} -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right) \overset{(2.5)}{=} \frac{mk}{L^2} u^0 \quad \Rightarrow \quad f\left(\frac{1}{u}\right) = -ku^2 \quad \Rightarrow \quad f(r) = -\frac{k}{r^2}$. The inverse square force

Observe that when $\beta = 2$, we have $J(0) \overset{(0)}{=} -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right) \overset{(2.5)}{=} \frac{mk}{L^2} u^{-3} \quad \Rightarrow \quad f\left(\frac{1}{u}\right) = -\frac{k}{u} \quad \Rightarrow \quad f(r) = -kr$. The radial harmonic oscillator force.

Hence, the only potentials that can produce stable closed non-circular orbits are the inverse-square force law ($\beta = 1$) and the radial harmonic-oscillator potential ($\beta = 2$). The solution $\beta = 0$ corresponds to perfectly circular orbits, as noted above.

3-Body Problem (Existence and Uniqueness):

The mathematical statement of the three-body problem can be given in terms of the Newtonian equations of motion for vector positions $r_i = (x_i, y_i, z_i)$ of three gravitationally interacting bodies with masses m_i:

$$\ddot{r}_1 = -Gm_2\frac{r_1-r_2}{|r_1-r_2|^3} - Gm_3\frac{r_1-r_3}{|r_1-r_3|^3}, \qquad \ddot{r}_2 = -Gm_3\frac{r_2-r_3}{|r_2-r_3|^3} - Gm_1\frac{r_2-r_1}{|r_2-r_1|^3}, \qquad \ddot{r}_3 = -Gm_2\frac{r_3-r_1}{|r_3-r_1|^3} - Gm_2\frac{r_3-r_2}{|r_3-r_2|^3},$$

where G is the gravitational constant. This is a set of 9 second-order differential equations. The problem can also be stated equivalently in the Hamiltonian formalism, in which case it is described by a set of 18 first-order differential equations, one for each component of the positions r_i and momenta p_i: $\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial r_i}$, where H is the Hamiltonian: $H = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3}\right) - G\left(\frac{m_1m_2}{|r_1-r_2|} + \frac{m_2m_3}{|r_3-r_2|} + \frac{m_3m_1}{|r_3-r_1|}\right)$. In this case H is simply the total energy of the system, gravitational plus kinetic.

Existence and Uniqueness of Solutions: We run into problems when there are collisions of the gravitationally attractive bodies causing singularities in the above differential equations.

Regularization Theory: Aim is to transform singular differential equations into regular ones, thus providing an efficient mathematical tool to analyze motions leading to collisions.

It has been shown that one can regularize double collisions, but not triple collisions. So, the solutions exist as long as we avoid triple collisions. Be wary of non-collision singularities!

Polar Coordinates:

Let $\vec{r} = (x,y) = r(\cos\varphi, \sin\varphi)$, so that $\vec{v} = \dot{r}(\cos\varphi, \sin\varphi) + r\dot{\varphi}(-\sin\varphi, \cos\varphi)$, and

$$\vec{a} = \ddot{r}(\cos\varphi, \sin\varphi) + 2\dot{r}\dot{\varphi}(-\sin\varphi, \cos\varphi) + r\ddot{\varphi}(-\sin\varphi, \cos\varphi) - r\dot{\varphi}^2(\cos\varphi, \sin\varphi). \qquad [0]$$

If we let $\hat{r} := (\cos\varphi, \sin\varphi)$, and $\hat{\varphi} := \frac{d\hat{r}}{d\varphi} = (-\sin\varphi, \cos\varphi)$ represent a new basis for our space (note $\hat{r}\cdot\hat{\varphi} = 0$), then we have $\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$, and $\vec{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{r} + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\varphi}$.

Calculating Equations of Motion - Newton

From Newton's **2nd Law of Motion** and **Law of Universal Gravitation**, we have $\vec{F} = m\vec{a} = -G\frac{Mm}{r^2}\hat{r}$ or $\vec{a} = -\frac{GM}{r^2}\hat{r}$. Substituting in [0], we have: $\left(\ddot{r} - r\dot{\varphi}^2\right)\hat{r} + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\varphi} = -\frac{GM}{r^2}\hat{r}$.

Separately equating the radial and angular components, gives us **equations of motion** (radial and angular, resp.):

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{GM}{r^2} \text{ and } 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \qquad [1a,b]$$

Note that the equations are coupled 2nd order ODEs. To understand what the equations mean, first we take a look at angular momentum: $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v}$. Using our velocity equation from above, we have: $\vec{L} = \vec{r}\times m\left(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}\right) = m\left(r\hat{r}\times\left(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}\right)\right)$, and noting that $\hat{r}\times\hat{\varphi} = \hat{k}$ and $\hat{r}\times\hat{r} = 0$, gives us: $\vec{L} = m\left(r^2\dot{\varphi}\right)\hat{k}$. And taking the derivative, we have: $\frac{d}{dt}\vec{L} = m\left(2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi}\right)\hat{k}$. And since for central forces, we have that the angular momentum is constant, we see that $2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$ (for nonzero radii) which is the 2nd of the equations of motion in [1]. So, [1] can be replaced with the requirement that the angular momentum is constant, and $L = mr^2\dot{\varphi}$. Solving for the rotation speed $\dot{\varphi}$, we have: $\dot{\varphi} = \frac{L}{mr^2}, \qquad [2]$

Substituting back into the first equation of [1], we have: $\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2}. \qquad [3]$

Thus, conservation of angular momentum "de-couples" the equations of motion. The equation has only one variable r, so it is often called a "one-dimensional equation." But you should always keep in mind that the motion takes place in two dimensions. Now we have the system in a form we can solve for, since [3] is an equation only in r. Once we

solve for r (using initial conditions) we can plug it into [2], and solve for φ, using initial conditions.

But how do we solve the differential equation $(1a)$?

Let $K := GM$ and $u := \frac{L^2}{Kr}$ or $r = \frac{L^2}{Ku}$.

Observe that $\dot{\varphi} = \frac{L}{mr^2} = \frac{K^2u^2}{mL^3}$, and using the chain rule: $\dot{r} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{d}{d\varphi}\left(\frac{L^2}{Ku}\right)\dot{\varphi} = -\frac{L^2}{Ku^2}\dot{\varphi}\frac{du}{d\varphi} = -\frac{K}{mL}\frac{du}{d\varphi}$.

Differentiating again, we obtain: $\ddot{r} = \frac{d\dot{r}}{d\varphi}\dot{\varphi} = -\frac{K}{mL}\frac{d^2u}{d\varphi^2}\dot{\varphi} = -\frac{K^3u^2}{m^2L^4}\frac{d^2u}{d\varphi^2}$.

Substituting into equation $1a$ $(\ddot{r} - r\dot{\varphi}^2 = -\frac{GM}{r^2})$ we obtain:

$$-\frac{K^3u^2}{m^2L^4}\frac{d^2u}{d\varphi^2} - \frac{L^2}{Ku}\left(\frac{K^2u^2}{mL^3}\right)^2 = -\frac{K}{\left(\frac{L^2}{Ku}\right)^2} = -\frac{K^3u^2}{L^4}.$$

or $\frac{d^2u}{d\varphi^2} + u = m^2$.

$r^2 + 1 = 0 \quad \Rightarrow \quad u_c = c_1\cos\varphi + c_2\sin\varphi$

$u_t = A \Rightarrow A = m^2 \Rightarrow u = u_p + u_c = m^2 + c_1\cos\varphi + c_2\sin\varphi$

$= m^2 + e\cos(\varphi - \varphi_0)$, where e and φ_0 are the constants of integration which can be determined from the initial conditions. In turn, we now know $r = \frac{L^2}{Ku} = \frac{L^2}{GM\left(m^2 + e\cos(\varphi - \varphi_0)\right)}$.

```
plot[t_] = NDSolve[{r"[t] - ((7^2)/(2^2*r[t])) == -((70)/(r[t]^2)), r[0] == 1, r'[0] == 8.31}, r[t], {t, 0, 1}][[1, 1, 2]];
Plot[plot[t], {t, 0, 4}]
```

Lagrangian Method

For an alternative method of calculating the **equations of motion**, we use the Lagrangian: $\mathcal{L} = T - U$, where U is the potential energy and T is the kenetic.

Recall that force can be defined as the negative of the vector gradient of the potential field: $\vec{F}\left(\vec{r}\right) = -\frac{dU}{d\vec{r}}$.

For a particle of mass m gravitationally attracted to a body of mass M :

$$U\left(\vec{r}\right) = -\int_{\infty}^{\vec{r}}\vec{F}\cdot d\vec{r} = -\int_{\infty}^{\vec{r}} -\frac{GMm}{r^2}\hat{r}\cdot d\vec{r} = -\frac{GMm}{r}.$$

The kinetic energy for this two-dimensional problem is $T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, or in polar coordinates: $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$. And the Lagrangian is: $\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GMm}{r}$.

Recall that the Lagrange **equations of motion** have the form: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0$, where the q_i's in our case are r and φ. So we have the two equations: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = 0$ and $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$. Taking the appropriate derivatives, this gives us: $\frac{d}{dt}\frac{\partial}{\partial\dot{r}}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{GMm}{r}\right) - \frac{\partial}{\partial r}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{GMm}{r}\right)$

$= \frac{d}{dt}\left(m\dot{r}\right) - mr\dot{\varphi}^2 + \frac{GMm}{r^2} = 0$, and

$$\frac{d}{dt}\frac{\partial}{\partial\dot{\varphi}}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{GMm}{r}\right) - \frac{\partial}{\partial\varphi}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{GMm}{r}\right) = \frac{d}{dt}\left(mr^2\dot{\varphi}\right) = \frac{dL}{dt} = 0.$$

This last equation gives us $\dot{\theta} = \frac{L}{mr^2}$. Substituting it into the first equation, we find: $\ddot{r} - \frac{L^2}{m^2r^3} + \frac{GM}{r^2} = 0$. Same as above, but much more easily computed than using Newton's 2nd law.

Hamiltonian of Newtonian N-Body Problem:

The n-body problem considers n point masses m_i, $i = 1, 2, \ldots, n$ in an inertial reference frame in three dimensional space $\mathbb{R}^3$ moving under the influence of mutual gravitational attraction. Each mass m_i has a position vector q_i.

Newton: His second law says mass times acceleration $m_i \frac{d^2 q_i}{dt^2}$ is equal to the sum of forces on the mass. Newton's law of gravity says gravitational force felt on mass m_i by a single mass m_j is given by: $F_{ij} = \frac{G m_i m_j (q_j - q_i)}{|q_j - q_i|^3}$ (metric induced by the l_2 norm). Summing over all masses yields the n-body equations of motion: $m_i \frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j (q_j - q_i)}{|q_j - q_i|^3} = -\frac{\partial U}{\partial q_i}$, where U is the self-potential energy: $U := -\sum_{1 \leq i < j \leq n} \frac{G m_i m_j}{|q_j - q_i|}$. A system of $3n$ second order ODEs, with $6n$ initial conditions as 3n initial position coordinates and 3n initial momentum values.

Hamilton: However, defining the momentum to be $p_i := m_i \frac{dq_i}{dt}$, we observe the kinetic energy is $T = \sum_{i=1}^{n} \frac{1}{2} m_i v^2 = \sum_{i=1}^{n} \frac{|p_i|^2}{2m_i}$, the Hamiltonian function is $H = T + U$, and Hamilton's equations of motion for the n-body problem are: $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$, $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$. Hamilton's equations show that the n-body problem is a system of 6n first-order differential equations.

Symmetries in the n-body problem yield global **integrals of motion** that simplify the problem. **Translational symmetry** of the problem results in the center of mass: $C = \frac{\Sigma m_i q_i}{\Sigma m_i}$, moving with constant velocity, so that $C = L_0 t + C_0$, where L_0 is the linear velocity and C_0 is the initial position. The constants of motion L_0 and C_0 represent six integrals of motion. **Rotational symmetry** results in the total angular momentum being constant: $A = \Sigma (q_i \times p_i)$. The three components of the total angular momentum A yield three more constants of motion. The last general constant of motion is given by the **conservation of energy** H. Hence, every n-body problem has ten integrals of motion.

Because T and U are homogeneous functions of degree 2 and -1, respectively, the equations of motion have a **scaling invariance**: if $q_i(t)$ is a solution, then so is $\lambda^{-\frac{2}{3}} q_i(\lambda t)$ for any $\lambda > 0$.

The **moment of inertia** of an n-body system is given by: $I = \Sigma m_i q_i \cdot q_i = \Sigma m_i |q_i|^2$, where q_i is a vector joining the body perpendicularly to the rotation axis. If the dynamics are planar, then this is just the normal distance vector. Then the **Lagrange–Jacobi formula** states that: $\frac{d^2 I}{dt^2} = 2T - U = T + h$, where h is the total energy of the system.

For systems in dynamic equilibrium, the longterm time average $\left\langle \frac{d^2 I}{dt^2} \right\rangle$ of I'' is zero. Then on average the total kinetic energy is half the total potential energy, $\langle T \rangle = \frac{1}{2} \langle U \rangle$, which is an example of the virial theorem for gravitational systems. If M is the total mass and R a characteristic size of the system (for example, the radius containing half the mass of the system), then the critical time for a system to settle down to a dynamic equilibrium is: $t_{cr} = \sqrt{\frac{GM}{R^3}}$.

Jacobi Coordinates (NBP):

In the theory of many-particle systems, Jacobi coordinates often are used to simplify the mathematical formulation. These coordinates are particularly common in treating polyatomic molecules and chemical reactions, and in celestial mechanics. An algorithm for generating the Jacobi coordinates for N bodies may be based upon binary trees. In words, the algorithm is described as follows:

> Let m_j and m_k be the masses of two bodies that are replaced by a new body of virtual mass $M = m_j + m_k$. The position coordinates x_j and x_k are replaced by their relative position $r_{jk} = x_j - x_k$ and by the vector to their center of mass $R_{jk} = (m_j q_j + m_k q_k)/(m_j + m_k)$. The node in the binary tree corresponding to the virtual body has m_j as its right child and m_k as its left child. The order of children indicates the relative coordinate points from x_k to x_j. Repeat the above step for $N - 1$ bodies, that is, the $N - 2$ original bodies plus the new virtual body.

For the N-body problem the result is: $r_j = \frac{1}{m_{0j}} \sum_{k=1}^{j} m_k x_k - x_{j+1}$, $r_N = \frac{1}{m_{0N}} \sum_{k=1}^{N} m_k x_k$, with $m_{0j} = \sum_{k=1}^{j} m_k$.

The vector r_N is the center of mass of all the bodies. The result one is left with is thus a system of $N - 1$ translationally invariant coordinates $r_1, \ldots, r_{N-1}$ and a center of mass coordinate r_N, from iteratively reducing two-body systems within the many-body system.

Symmetries ↔ Conserved Quantities (NBP):

Given a collection of point-particles, interacting through an attractive force $\sim \frac{1}{r^2}$. Knowing only $m_1 a_i = \sum_i \frac{G m_1 m_i}{r_i^2}$ and initial conditions we can observe that three quantities remains constant: A) center of mass of the system, B) total energy, and C) angular momentum.

A) Center of mass is the most basic, it needs just Newton's laws:
Second Law: $m_i \ddot{\vec{r}}_i = F_i$, and Third Law: $\sum_i F_i = 0$.
Summing over *i* one obtains: $\frac{d^2}{dt^2}(\sum_i m_i r_i) = \sum_i F_i = 0$.
So we have $\frac{\sum_i m_i r_i}{\sum_i m_i} = c_1 t + c_2$, but by the symmetry of translation invariance (conservation of linear momentum), we can choose a moving inertial reference frame such that $\frac{\sum_i m_i r_i}{\sum_i m_i} = 0$.

B) For the total energy you need the force/potential to be independent of time:
$F_i = -\frac{d}{dr_i} U(r_1, r_2, \ldots)$. Then just take the total energy : $E = \sum_i \frac{m_i |\dot{r}_i|^2}{2} + U$.
And differentiate with respect to time: $\frac{dE}{dt} = \sum_i m_i (\ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i) + \sum_i \frac{dU}{dr_i} \dot{\vec{r}}_i = (F - F) \cdot \dot{\vec{r}}_i = 0$.

C) Finally, for the angular momentum, *U* must have the symmetry of rotational invariance:
$U(R\vec{r}_1, R\vec{r}_2, R\vec{r}_3, \ldots) = U(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots)$, where *R* is a rotation matrix. Consider now very small (infinitesimal) rotation: $R\vec{r}_i = \vec{r}_i + [\delta\varphi \times \dot{r}_i]$ for angle $\delta\varphi$. Substituting and expanding, one can get: $U + \delta\varphi \sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = U$. Which works for every angle $\delta\varphi$, so the following must hold $\sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = 0$. Finally take the angular momentum: $L = \sum_i m_i [\vec{r}_i \times \dot{r}_i]$, and differentiate with respect to time:
$\frac{dL}{dt} = \sum i [\vec{r}_i \times m_i \ddot{\vec{r}}_i] + \sum_i m_i [\dot{r}_i \times \dot{r}_i] = \sum_i [-m_i \ddot{\vec{r}}_i \times \vec{r}_i] + 0 = \sum_i [-F_i \times \vec{r}_i] = \sum_i [\frac{dU}{d\vec{r}_i} \times \vec{r}_i] = 0$.

Levi-Civita Transformation (**NBP**): http://assets.press.princeton.edu/chapters/s7687.pdf

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Central Configurations of the Newtonian n-Body Problem

Central Force: Suppose that a force acting on a particle of mass *m* has the properties:

- The force is always directed from *m* toward, or away, from a fixed point *O*,
- The magnitude of the force only depends on the distance *r* from *O*.

Forces having these properties are called central forces.

Mathematically, $\vec{F}$ is a central force if and only if: $\vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}$, where $\hat{r} = \frac{\vec{r}}{r}$ is a unit vector in the direction of $\vec{r}$, (the position vector of the particle).

If $f(r) < 0$, the force is said to be attractive, and if $f(r) > 0$, the force is said to be repulsive.

Properties of a particle moving under the influence of a central force:

1. The path of the particle must be a **plane curve**
2. The angular momentum of the particle is conserved, i.e., it is constant in time.
3. The particle moves in such a way that the position vector sweeps out equal areas in equal times.

In other words, the time rate of change in area is constant. This is referred to as the **Law of Areas**.

Angular Momentum is Constant / Central Force Motion is Planar

Let's show that the motion of a particle under a central force $\vec{F}$ always remains in the plane defined by its initial position and velocity. We show that $\vec{F}$ lies in this plane. And since the position $\vec{r}$, velocity $\vec{v}$ and force $\vec{F}$ all lie in the same plane, there is never an acceleration perpendicular to that plane. To this end, it suffices to show that the

angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of the particle is constant, where $\vec{p}$ is its linear momentum. If $\vec{F}$ were not in the plane, then this would imply that the force would move $\vec{r}, \vec{v}$ to form a new plane, and $\vec{L}$, being perpendicular to the new plane, would change. Recall that the time derivative of the angular momentum is:

$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\vec{v} \times m\vec{v}) + \left(\vec{r} \times \frac{d}{dt}\vec{p}\right) = \vec{r} \times \vec{F} =$ net torque.

When $\vec{F}$ is a central force, the remaining term $\vec{r} \times \vec{F}$ is also zero because the vectors $\vec{r}$ and $\vec{F}$ point in the same or opposite directions. Therefore, the angular momentum $\vec{L}$ is constant:

$\frac{d\vec{L}}{dt} = \vec{r} \times f(r)\hat{r} = r\hat{r} \times f(r)\hat{r} = rf(r)(\hat{r} \times \hat{r}) = 0.$

Another way of looking at this is to recall that the time rate of change of angular momentum is equal to the torque. A central force cannot exert a torque on a particle. These simple physical arguments lead us to an important conservation law concerning the motion of any particle under the action of any central force: both the magnitude and the direction of the angular momentum are constant. Also note that:
$\vec{r} \cdot \vec{L} = \vec{r} \cdot \left(\vec{r} \times \vec{p}\right) = \vec{p} \cdot \left(\vec{r} \times \vec{r}\right) = 0.$ Consequently, the particle's position $\vec{r}$ (and hence velocity $\vec{v}$) always lies in a plane perpendicular to $\vec{L}$.

Central Force Fields are Conservative, Computing the Potential

Proof: Since we have a central force: $F = f(r)\frac{\vec{r}}{r} =: g(r)\vec{r}$. Calculating the curl: $\nabla \times F = \nabla \times \left(g\vec{r}\right)$.

Recall: $\nabla \times (c\vec{v}) = \nabla c \times \vec{v} + c(\nabla \times \vec{v})$.

So, $\nabla \times \left(g\vec{r}\right) = \nabla g \times \vec{r} + g\left(\nabla \times \vec{r}\right) = \nabla g \times \vec{r} + 0 = \frac{dg}{dr}\left(\hat{r} \times \vec{r}\right) = 0.$

Therefore, the curl is zero, and the central force is conservative.

Another Source

To be considered conservative, the work done by a force $\vec{F}$ in moving a particle between two points must be independent of the path taken between the two points, i.e., it only depends on the endpoints of the path.

Potential energy is closely linked with forces. If the work done by a force on a body that moves from A to B does not depend on the path between these points (if the work is done by a conservative force), then the work of this force measured from A assigns a scalar value to every other point in space and defines a scalar potential field. In this case, the force can be **defined** as the negative of the vector gradient of the potential field. $\vec{F}\left(\vec{r}\right) = -\nabla V = -\frac{dV}{d\vec{r}}$, which implies $V\left(\vec{r}_0\right) = -\int_{\infty}^{\vec{r}_0} \vec{F}\left(\vec{r}\right) \cdot d\vec{r}$ (Note that $\vec{F}(\infty) = 0$). So, the work done is:
$W\left(\vec{r}\right) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = V\left(\vec{r}_1\right) - V\left(\vec{r}_2\right) = -\nabla V$. Assume $\vec{r}_1 \to \infty$, then $W(\vec{r}) = -V(\vec{r})$ (scalar potential field).
Therefore, the work done depends only on the difference between the potential energy at the final position and the initial position, is independent of the path, and a central force is a conservative force. One consequence is that total energy is conserved.

But how do we compute the potential in a **central force field**? Note: $\vec{F}\left(\vec{r}\right) = -\frac{dV}{dr}$ implies $\vec{F} \cdot d\vec{r} = -dV.$ [1]
Observe that $\vec{r} \cdot \vec{r} = r^2$, and $d\left(\vec{r} \cdot \vec{r}\right) = d(r^2) \quad \Rightarrow \quad \left(\vec{r} \cdot d\vec{r}\right) + \left(d\vec{r} \cdot \vec{r}\right) = 2rdr$. So we have: $\vec{r} \cdot d\vec{r} = rdr$.
So evaluating the left-hand side of [1], we have: $\vec{F} \cdot d\vec{r} = f(r)\frac{\vec{r}}{r} \cdot d\vec{r} = f(r)dr$. Therefore, $f(r)dr = -dV$, from which it follows that: $V = -\int f(r)dr$. Hence, if we know the central force field $\vec{F} = f(r)\frac{\vec{r}}{r}$, this equation tells us how to compute the potential.

Specific Angular Momentum: The angular momentum divided by the mass, $\vec{h} := \frac{\vec{L}}{m}$, and

$$h := \frac{|\vec{L}|}{m} = r^2\ \dot{\varphi}\ = \left|\vec{r} \times \vec{v}\right|.$$

Constant Areal Velocity "Law of Areas"

Suppose that in time Δt, the position vector moves from $\vec{r}$ to $\vec{r} + \Delta\vec{r}$.

Claim: The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\vec{r}$ and $\Delta\vec{r}$.

Recall the Area of Parallelogram = $(height) \cdot \left|\vec{r}\right| = \left|\Delta\vec{r}\right| \sin\theta \left|\vec{r}\right| = \left|\vec{r} \times \Delta\vec{r}\right|$, where θ is the angle between $\vec{r}$ and $\Delta\vec{r}$. So, $\Delta A := \frac{1}{2}\left|\vec{r} \times \Delta\vec{r}\right|$.

One notes visually, that this shape roughly approximates the shape we are interested in, and as $\Delta t \to 0$, this approximation becomes exact. Dividing by Δt, and letting $\Delta t \to 0$, gives us:
$\dot{A} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}\left|\vec{r} \times \frac{\Delta r}{\Delta t}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}\right|$. To evaluate $\vec{r} \times \vec{v}$, we use $\vec{r} = r\hat{r}$, and we have:
$\vec{r} \times \vec{v} = \vec{r} \times \left(\dot{r}\ \hat{r} + r\ \dot{\theta}\ \hat{\theta}\right) = r\ \dot{r}\ (\hat{r} \times \hat{r}) + r^2\ \dot{\theta}\ \left(\hat{r} \times \hat{\theta}\right) = r^2\ \dot{\theta}\ \vec{k}$. Therefore, we have (via the specific angular momentum): $2\ \dot{A} = \left|\vec{r} \times \vec{v}\right| = r^2\ \dot{\theta} = h = \frac{|\vec{L}|}{m}$ = constant, and the vector: $\dot{\vec{A}} = \dot{A}\ \vec{k} = \frac{1}{2} r^2\ \dot{\theta}\ \vec{k} = \frac{h}{2}\vec{k}$, is called the **areal velocity**.

Regularization Theory

As the two bodies approach each other, their distance tends to zero and, consequently, the differential equation describing the dynamics of the system becomes singular when the two bodies collide. In the real world the point masses are replaced by bodies of finite (non zero) size; in this case, we refer to a binary collision whenever the distance between their centers of mass equals the sum of the radii. Even in the simplified approach that solar system objects are reduced to point masses, the description of the dynamics of an N–body system becomes difficult during close encounters, due to the loss of regularity of the EOM.

In order to get rid of this problem, regularization theories have been developed to transform a singular equation into a regular one. An important consequence of this property is the continuous dependence of the entire orbit upon initial conditions, even in the presence of binary collisions. An additional and important benefit of Levi-Civita's approach is that the regularized equations of motion of the unperturbed Kepler problem are linear. This allows the development of simple perturbation theories.

One such method is the so–called Levi–Civita regularization, which is based on an ad hoc transformation of the spatial variables and of time (through the introduction of a fictitious time), taking advantage of the conservation of energy. The description of the motion fails at the singularity, but what is even worse, it is rather difficult to investigate the dynamics in a neighbourhood of the singularity.

Even excluding a collision, it is troublesome to explore the trajectories corresponding to close approaches: the integration step required in a numerical approach is usually very small, thus requiring a huge amount of computational time. The Levi–Civita regularization, is based upon three main steps:

- the introduction of a suitable change of coordinates (typically the Levi–Civita transformation);
- a stretching of the time scale, to get rid of the fact that the velocity becomes infinite (namely, the introduction of a so–called fictitious time);
- the conservation of the energy, to transform the singular differential equations into regular ones (i.e., the study of the Hamiltonian system in the extended phase space).

Starting with $\left|\vec{f}\right| = \frac{mm_p}{r^2}$, we assume m_p to be negligible, and m to be situated at the origin. We find the equation of

motion $\ddot{r} = -m\frac{1}{r^2}$, which has solutions $r = ct^{\alpha}$. If $\alpha = \frac{2}{3}$. We determine $c = \left(\frac{9}{2}m\right)^{\frac{1}{3}}$. This motion is one-dimensional parabolic, with a singularity at $t = 0$ of algebraic type $r = O\left(t^{\frac{2}{3}}\right)$. The continuation corresponds to the analytic continuation of $r(t)$ onto a different sheet of the Riemann surface of the corresponding analytic function.

Levi-Civita Regularization

Let $x = (x_1, x_2)$. Or $x = x_1 + ix_2$, with $\overline{x} = x_1 - ix_2$. Our EOM become: $\ddot{x} = M\frac{x}{r^3} + f(x,t)$, where $M = m + m_p$ and $r = |x|$. (4)

f represents the perturbative force vector, the pure Kepler motion being obtained with $f = 0$. The energy function is:
$-H = \frac{1}{2}\left|\dot{x}\right|^2 - \frac{M}{r}$, (5)

where H is the negative energy satisfying the differential equation and initial condition: $\dot{H} = -\left\langle \dot{x}, f \right\rangle$, and $H(0) = \frac{M}{|x(0)|} - \frac{1}{2}\left|\dot{x}\ (0)\right|^2$. The sign of H has been chosen such that $H > 0$ corresponds to elliptic motion in the unperturbed Kepler problem.

Step 1 — Time Transformation (to eliminate infinite velocity)

Observe from (5) that $\dot{x} = \pm\sqrt{2(\frac{m}{r} - H)}$ is ∞ when $r = 0$. This complicates our analysis, so let us transform time to illuminate this complication.

Let: $dt = rd\tau$. Therefore, the ratio $\frac{dt}{d\tau}$ of the two infinitesimal increments is made proportional to the distance r, the movie is run slow motion whenever r is small. This will also add a variable to our system and our DEQ, giving us an extended phase space. Observe that $\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{x'}{r}$. Substituting into our velocity equation, we find a finite velocity $x' = \sqrt{2(Mr - Hr^2)}$ is finite for small r. Substituting into the EOM and H ((4) and (5)):
$rx'' - r'x' + Mx = r^3 f$, and $\frac{1}{2r^2}|x'|^2 - \frac{M}{r} = -H$. (8)

Step 2 — Conformal Squaring (Levi-Civita Transformation)

Represent the complex physical coordinate x as u^2 of $u = u_1 + iu_2$. So, $x = u^2$ (9)

The parametric u-manifold is a Riemann surface with two sheets, connected by branch points at $u = 0$ and $u = \infty$. From (9), we have: $r = |x| = |u|^2 = u\overline{u}$, (10)

and differentiating (9) and (10) with respect to τ: $x' = 2uu'$, $x'' = 2\left(uu'' + (u')^2\right)$, and
$r' = u'\overline{u} + u\overline{u}'$. (11)

Substituting into the EOM and H with (8):

$2r\left(uu'' + (u')^2\right) - (u'\overline{u} + u\overline{u}')(2uu') + Mu^2 = r^3 f$, and $\frac{1}{2r^2}|2uu'|^2 - \frac{M}{r} = -H$

$2ruu'' + Mu^2 + 2|u^2|(u')^2 - 2u\overline{u}u'u' - 2u^2u'\overline{u}' = r^3 f$, and $\frac{2}{r}|uu'|^2 - M = -rH$

$2ruu'' + Mu^2 - 2u^2u'\overline{u}' = r^3 f$, and $M - \frac{2}{|u^2|}|uu'|^2 = rH$

$2ruu'' + u^2\left(M - 2|u'|^2\right) = r^3 f$, and $M - 2|u'|^2 = rH$

$u'' + \frac{1}{2}u\left(\frac{M}{u^2} - 2\frac{|u'|^2}{u^2}\right) = \frac{r^2}{u}\frac{f}{2}$, and $\frac{M}{u^2} - 2\frac{|u'|^2}{u^2} = H$. (12)

Step 3 — Elimination of First Derivatives (to eliminate singularities in DEQ)

Produces linear DEQs for the unperturbed problem $f = 0$. Substituting the 2nd equation of (12) into the first equation, we have: $u'' + \frac{1}{2}uH = rf\overline{u}$ (using (10)).

Theorem: The perturbed Kepler problem (4) with energy equation (5) is equivalent to:

$$2u'' + uH = r^2 f\tfrac{1}{u} = rf\overline{u}, \qquad t' = r, \qquad H' = \langle x', f\rangle, \text{ with } H(0) = \tfrac{M}{|x(0)|} - \tfrac{1}{2}\left|\dot{x}\ (0)\right|^2,$$

DEQs for the dependent variables u, t, H as functions of fictitious time τ.

Regularization has now been accomplished: all collisions, i.e. all passages of u through 0 have no singularities in the solution as long as the perturbation is regular.

Relative Equilibrium

A relative equilibrium in a symmetric dynamical system is a **group orbit** that is invariant under the dynamics. Another way of viewing a relative equilibrium is to consider the dynamics induced on the orbit space (system after quotienting out of the orbit angle) of the phase space, and then a relative equilibrium is just an equilibrium point of this induced dynamics. For finite groups, relative equilibria are just (group orbits of) equilibria.

Relative equilibria are the symmetric analogues of fixed equilibria for dynamical systems which are invariant under the action of a Lie group. Examples include circular orbits in the Kepler problem, rigid bodies and heavy tops rotating about an axis of symmetry, and rotating systems of point vortices with constant geometric configurations.

A relative equilibrium of a Hamiltonian system with symmetry is a point of phase space giving an evolution which is a one-parameter orbit of the action of the symmetry group of the system. The evolutions of sufficiently small perturbations of a formally stable relative equilibrium are arbitrarily confined to that relative equilibrium's orbit under the isotropy subgroup of its momentum.

Given the correct initial velocities, a central configuration will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.

Relative equilibria and relative periodic solutions are related by symmetry reduction to equilibria and periodic solutions of the reduced dynamics.

A relative equilibrium is a solution which travels along an orbit of the symmetry group at constant speed.

Relative equilibria of the Nbody problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. A striking application of relative periodic orbits has been the discovery of "choreographies" in the N-body problems.

For the planar N-body problem, relative equilibria are uniformly rotating rigid solutions, and therefore, such configurations have a moment of inertia that is constant in time.

The modern story on equivariance and dynamical systems starts perhaps with S. Smale [10.26] and M. Field [10.27], and on bifurcations in presence of symmetries with Ruelle [10.28]. Ruelle proves that the stability matrix/Jacobian matrix evaluated at an equilibrium/fixed point $x \in \boldsymbol{\mathcal{M}}_G$ decomposes into linear irreducible representations of G, and that stable/unstable manifold continuations of its eigenvectors inherit their symmetry properties, and shows that an equilibrium can bifurcate to a rotationally invariant periodic orbit (i.e., relative equilibrium).

Relative Equilibrium ⇔ "Critical points of the amended potential"

For symmetry group G, if we have a group action $g \in G$ (in our case a rotation). Dynamics f^t, in our case the equations of motion. Then, for an invariant group orbit, we have $g\left(f^t\right) = f^t(g)$, or the group action commutes with the dynamics. Which implies that the equations of motion are invariant under the group action. *Given* the factor field $\vec{v}(x)$, then we have $\vec{v}(x) = g^{-1} \circ \vec{v} \circ g$, in other words, the vector field (equation of motions) are invariant under the group action. Said in another way, if our state space manifold is M, then (M,f) is G-invariant, also called G-equivariant.

Stability of Relative Equilibria

Every planar CC determines a relative equilibrium solution of the N-body problem with every body moving on a circle with constant angular speed. In a uniformly rotating coordinate system these become equilibrium solutions and one can analyze their linear stability by finding the eigenvalues of the linearized differential equations. Strictly speaking, even the circular periodic orbits of the two-body problem are linearly unstable because they are part of a four-parameter family of RE solutions with different sizes, centers of mass and rotation speeds. This applies to all relative equilibria and is reflected in the existence of four eigenvalues with value 0 organized into two Jordan blocks. This trivial drifting apart of nearby relative equilibria can be eliminated by fixing the center of mass, energy and angular momentum. Then for linear stability it is necessary that the rest of the eigenvalues be purely imaginary numbers.

With this understanding, it is known (Gascheau (1842)) that Lagrange's equilateral triangle solutions are linearly stable provided the masses satisfy: $27(m_1m_2 + m_3m_1 + m_2m_3) < (m_1 + m_2 + m_3)^2$.

This holds only if there is a dominant mass, i.e., one mass must be much larger than the other two. It does hold in the case of the Sun, Jupiter and a third small body so, ignoring the other planets, a small body forming an equilateral triangle with the Sun and Jupiter could remain there for a long time. In fact there are clusters of small bodies found near these equilateral points: the Greek and Trojan asteroids. On the other hand, the collinear Eulerian relative equilibria are always linearly unstable.

The RE based on the regular N-gon with N equal masses is linearly unstable as is the centered (additional mass at the center) N-gon with $N + 1$ equal masses. On the other hand a regular N-gon with a sufficiently large $(N + 1)$-st mass at the center is linearly stable, provided N is at least 7 (Moeckel (1995)). For small N, lumpy rings can be linearly stable.

James Clerk Maxwell studied the centered N-gon in connection with the stability of the rings of Saturn. He found that the mass ratio guaranteeing stability increases with N, so a given ring mass should not be split into too many pieces. This supported the idea that the rings were composed of discrete bodies instead of a continuously distributed dust or liquid. (Maxwell (1859)).

2BP wikipedia.org/wiki/Two-body_problem

Circular orbits of the 2 body problem are the simplest examples.

The two-body problem is to predict the motion of two massive objects which are abstractly viewed as point particles. The problem assumes that the two objects interact only with one another; the only force affecting each object arises from the other one, and all other objects are ignored.

The most prominent case of the classical two-body problem is the gravitational case (see also Kepler problem), arising in astronomy for predicting the orbits (or escapes from orbit) of objects such as satellites, planets, and stars. A two-point-particle model of such a system nearly always describes its behavior well enough to provide useful insights and predictions.

A simpler "one body" model, the "central-force problem", treats one object as the immobile source of a force acting on other. One then seeks to predict the motion of the single remaining mobile object. Such an approximation can give useful results when one object is much more massive than the other (as with a light planet orbiting a heavy star, where the star can be treated as essentially stationary).

However, the one-body approximation is usually unnecessary except as a stepping stone. For many forces, including gravitational ones, the general version of the two-body problem can be reduced to a pair of one-body problems, allowing it to be solved completely, and giving a solution simple enough to be used effectively.

By contrast, the three-body problem (and, more generally, the n-body problem for $n \geq 3$) cannot be solved in terms of first integrals, except in special cases.

Gravitation and other inverse-square examples

The two-body problem is interesting in astronomy because pairs of astronomical objects are often moving rapidly in

arbitrary directions (so their motions become interesting), widely separated from one another (so they won't collide) and even more widely separated from other objects (so outside influences will be small enough to be ignored safely).

Under the force of gravity, each member of a pair of such objects will orbit their mutual center of mass in an elliptical pattern, unless they are moving fast enough to escape one another entirely, in which case their paths will diverge along other planar conic sections. If one object is very much heavier than the other, it will move far less than the other with reference to the shared center of mass. The mutual center of mass may even be inside the larger object.

In principle, the same solutions apply to macroscopic problems involving objects interacting not only through gravity, but through any other attractive scalar force field obeying an inverse-square law, with electrostatic attraction being the obvious physical example. In practice, such problems rarely arise. Except perhaps in experimental apparatus or other specialized equipment, we rarely encounter electrostatically interacting objects which are moving fast enough, and in such a direction, as to avoid colliding, and/or which are isolated enough from their surroundings.

Inapplicability to atoms and subatomic particles

Although the two-body model treats the objects as point particles, classical mechanics only apply to systems of macroscopic scale. Most behavior of subatomic particles cannot be predicted under the classical assumptions underlying this article or using the mathematics here.

Electrons in an atom are sometimes described as "orbiting" its nucleus, following an early conjecture of Niels Bohr (this is the source of the term "orbital"). However, electrons don't actually orbit nuclei in any meaningful sense, and quantum mechanics are necessary for any useful understanding of the electron's real behavior. Solving the classical two-body problem for an electron orbiting an atomic nucleus is misleading and does not produce many useful insights.

Reduction to two independent, one-body problems

The complete two-body problem can be solved by re-formulating it as two one-body problems: a trivial one and one that involves solving for the motion of one particle in an external potential. Since many one-body problems can be solved exactly, the corresponding two-body problem can also be solved.

Let x_1 and x_2 be the vector positions of the two bodies, and m_1 and m_2 be their masses. The goal is to determine the trajectories $x_1(t)$ and $x_2(t)$ for all times t, given the initial positions $x_1(t = 0)$ and $x_2(t = 0)$ and the initial velocities $v_1(t = 0)$ and $v_2(t = 0)$.

When applied to the two masses, Newton's second law states that: $\vec{F}_{12}(x_1,x_2) = m_1 \ddot{x}_1$, [1]

and $\vec{F}_{21}(x_1,x_2) = m_2 \ddot{x}_2$, [2]

where F_{12} is the force on mass 1 due to its interactions with mass 2, and F_{21} is the force on mass 2 due to its interactions with mass 1.

Adding and subtracting these two equations decouples them into two one-body problems, which can be solved independently. Adding equations (1) and (2) results in an equation describing the center of mass (barycenter) motion. By contrast, subtracting equation (2) from equation (1) results in an equation that describes how the vector $r = x_1 - x_2$ between the masses changes with time. The solutions of these independent one-body problems can be combined to obtain the solutions for the trajectories $x_1(t)$ and $x_2(t)$.

Center of mass motion (1st one-body problem)

Let $\vec{R} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ be the position of the center of mass (**barycenter**) of the system. Addition of the force equations (1) and (2) yields: $m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = (m_1 + m_2) \ddot{R} = \vec{F}_{12} + \vec{F}_{21} = 0,$

where we have used Newton's third law $F_{12} = -F_{21}$ and where

$\ddot{R} = \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2}$.

The resulting equation: $\ddot{R} = 0$ shows that the velocity $V = dR/dt$ of the center of mass is constant, from which follows that the **total momentum** $m_1 v_1 + m_2 v_2$ is also constant (**conservation of momentum**). Hence, the position

$R(t)$ of the center of mass can be determined at all times from the initial positions and velocities.

Displacement vector motion (2nd one-body problem)

Dividing both force equations by the respective masses, subtracting the second equation from the first and rearranging gives the equation: $\ddot{r}=\ddot{x}_1 \ -\ddot{x}_2 \ = \left(\frac{F_{12}}{m_1} - \frac{F_{21}}{m_2}\right) = (\frac{1}{m_1} + \frac{1}{m_2})F_{12}$ where we have again used Newton's third law $F_{12} = -F_{21}$ and where r is the displacement vector from mass 2 to mass 1, as defined above.

The force between the two objects, which originates in the two objects, should only be a function of their separation r and not of their absolute positions x_1 and x_2; otherwise, there would not be translational symmetry, and the laws of physics would have to change from place to place. The subtracted equation can therefore be written: $\mu \ \ddot{r} \ = F_{12}(x_1, x_2) = \vec{F}\left(\vec{r}\right)$ where μ is the reduced mass $\mu = \frac{1}{\frac{1}{m_1}+\frac{1}{m_2}} = \frac{m_1 m_2}{m_1+m_2}$. Solving the equation for $r(t)$ is the key to the two-body problem. The solution depends on the specific force between the bodies, which is defined by $\vec{F}\left(\vec{r}\right)$. For the case where $\vec{F}\left(\vec{r}\right)$ follows an inverse-square law, see the Kepler problem.

Once $R(t)$ and $r(t)$ have been determined, the original trajectories may be obtained: $\vec{x}_1(t) = \vec{R}(t) + \frac{m_2}{m_1+m_2}\vec{r}(t)$, $\vec{x}_2(t) = \vec{R}(t) - \frac{m_1}{m_1+m_2}\vec{r}(t)$ as may be verified by substituting the definitions of R and r into the right-hand sides of these two equations.

Gravitational two-body problem

The reduced mass multiplied by the relative acceleration between the two bodies is equal to the gravitational force ($\vec{F} = \frac{m_1 m_2}{m_1+m_2}\vec{a}_r = M\vec{a}_r$). The latter is proportional to the product of the two masses, which is equal to the reduced mass divided by the sum of the masses. Thus in the differential equation the two occurrences of the reduced mass cancel each other, and we get the same differential equation as for the position of a very small body orbiting a body with a mass equal to the sum of the two masses.

Assume: the vector $\vec{r}$ is the position of one body relative to the other

- $\vec{r}, \vec{v} \equiv \frac{d\vec{r}}{dt}$, the **semi-major axis** a, and the **specific relative angular momentu**m h are defined accordingly (hence r is the distance)
- h (**specific relative angular momentu**m) is the total **angular momentum** divided by the reduced mass $\frac{L}{M}$
- $\mu = G(m_1 + m_2)$, the standard gravitational parameter (the sum of those for each mass) where: m_1 and m_2 are the masses of the two bodies.

Then:

- the general solution is (see also the orbit equation and two-body problem for an inverse-square force): $u(\theta) \equiv \frac{1}{r(\theta)} = \frac{\mu}{h^2}(1 + e\cos(\theta - \theta_0))$ for any nonnegative e, called the eccentricity; here θ is the **true anomaly**, which is the angle between the current position of the orbiting object and the location in the orbit at which it is closest to the central body (called the periapsis).
- the positions of the bodies with respect to the barycenter are $\frac{m_2}{m_1+m_2}$ and $\frac{m_1}{m_1+m_2}$ times r, respectively, so the two bodies' orbits are similar conic sections; the same ratios apply for the velocities, and, without the minus, for the angular momentum and for the kinetic energies, all with respect to the barycenter
- for circular orbits $rv^2 = r^3\omega^2 = \frac{4\pi^2 r^3}{T^2} = \mu$
- for elliptic orbits: $\frac{4\pi^2 a^3}{T^2} = \mu$ (with a expressed in AU and T in years, and with M the total mass relative to that of the Sun, we get $\frac{a^3}{T^2} = M$)
- for parabolic trajectories rv^2 is constant and equal to 2μ
- the formulas for the specific orbital energy ε apply, with specific potential and kinetic energy and their sum taken as the totals for the system, divided by the reduced mass; the kinetic energy of the smaller body is larger; the potential energy of the whole system is equal to the potential energy of one body with respect to the other, i.e. minus the energy needed to escape the other if the other is kept in a fixed position; this should not be confused with the smaller amount of energy one body needs to escape, if the other body moves away also, in the opposite direction: in that case the total energy the two need to escape each other is the same as the

aforementioned amount; the conservation of energy for each mass means that an increase of kinetic energy is accompanied by a decrease of potential energy, which is for each mass the inner product of the force and the change in position relative to the barycenter, not relative to the other mass

- for elliptic and hyperbolic orbits $\mu = 2a|\varepsilon|$

For example, consider two bodies like the Sun orbiting each other:

- the reduced mass is one half of the mass of one Sun (one quarter of the total mass) at a distance of 1 AU: the **orbital period** is $\frac{\sqrt{2}}{2}$ year, the same as the orbital period of the Earth would be if the Sun had twice its actual mass; the total energy per kg reduced mass (90 MJ/kg) is twice that of the Earth–Sun system (45 MJ/kg); the total energy per kg total mass (22.5 MJ/kg) is one half of the total energy per kg Earth mass in the Earth–Sun system (45 MJ/kg)
- at a distance of 2 AU (each following an orbit like that of the Earth around the Sun): the orbital period is 2 years, the same as the orbital period of the Earth would be if the Sun would have one quarter of its actual mass at a distance of $\sqrt[3]{2} \approx 1.26$ AU: the orbital period is 1 year, the same as the orbital period of the Earth around the Sun. Similarly, a second Earth at a distance from the Earth equal to $\sqrt[3]{2}$ times the usual distance of geosynchronous orbits would be geosynchronous.

Central Configuration

A central configuration is a special arrangement of point masses interacting by Newton's law of gravitation with the following property: the gravitational acceleration vector produced on each mass by all the others should point toward the center of mass and be proportional to the distance to the center of mass. Central configurations (or CC's) play an important role in the study of the Newtonian N-body problem. For example, they lead to the only explicit solutions of the equations of motion, they govern the behavior of solutions near collisions, and they influence the topology of the integral manifolds.

The Basic CC Equations

Newton's law of motion for the gravitational N-body problem is: $m_i \ddot{q}_i = F_i = \sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{r_{ij}^3}$ (1)

Where $r_{ij} = |q_i - q_j|$. The force vector $F_i \in \mathbb{R}^d$ on the right-hand side can also be written as a partial gradient vector $F_i = \nabla_i U$ where: $U = \sum_{j \neq i} \frac{m_i m_j}{r_{ij}}$ (2)

is the Newtonian potential function and ∇_i denotes the vector of partial derivatives with respect to the d components of q_i . The acceleration of the i-th body is $\frac{F_i}{m_i}$ so the condition for q to be a central configuration is:

$$\nabla_i U = -\lambda m_i (q_i - c) \quad (3)$$

where $c = \frac{m_1 q_1 + \ldots + m_N q_N}{m_1 + \ldots + m_N} \in \mathbb{R}^d$ (4)

is the center of mass and $\lambda \in R$ is a constant. By definition, $q \in \mathbb{R}_N^d$ is a central configuration for the masses m_i if and only if (3) and (4) hold for some constant λ.

It turns out, however, that **the values of λ and c are uniquely determined by (3)**.

> **Proof**: Note that by translation invariance and degree -1 homogeneity of the Newtonian potential we have:
> $\sum_i \nabla_i U = 0, \quad \sum_i q_i^T \nabla_i U = -U.$
> Together these give $\sum_i (q_i - c)^T \nabla_i U = -U$ and then (3) shows that $\lambda = U/I$, where: $I = \sum m_i |q_i - c|^2$ (5)
> is the moment of inertia with respect to c.
> Thus $\lambda > 0$ is uniquely determined. Finally, summing (3) shows that c must be the center of mass.

Thus q is a central configuration for the given masses if and only if (3) holds for some (unique): $\lambda \in \mathbb{R}, \ c \in \mathbb{R}^d$.

Equivalent Central Configurations and Normalized Equations

The central configuration equation (3) is invariant under the Euclidean similarities of $\mathbb{R}^d$ – translations, rotations, reflections and dilations. Call two configurations $q, q' \in \mathbb{R}^{Nd}$ equivalent if there are constants $k \in \mathbb{R}, \ b \in \mathbb{R}^d$ and a $d \times d$ orthogonal matrix Q such that $q_i' = kQq_i + b, \ i = 1, \ldots, N$. If q satisfies (3) with constants λ, c then q' satisfies (3) with constants $\lambda' = k^3 \lambda, \ c' = c + b$. So one can speak of an equivalence class of central configurations.

Translation invariance can be used to eliminate the center of mass. For configurations with $c = 0$ the central configuration equations become: $-\lambda q_i = \sum_{j \neq i} \frac{m_j(q_j - q_i)}{r_{ij}^3}$ (6)

and any configuration satisfying this equation has $c = 0$. Alternatively, substituting (4) into (3) leads, after some simplification, to: $\sum_{j \neq i} m_j S_{ij}(q_j - q_i) = 0, \quad S_{ij} = \frac{1}{r_{ij}^3} - \lambda'$ (7)

where $\lambda' = \frac{\lambda}{m_1 + \ldots + m_N}$.

Dilation invariance can be used to normalize the size of a central configuration. The moment of inertia (5) is a natural measure of the size and setting $I = 1$ is a popular normalization. Alternatively, one can normalize the size by fixing the value of λ in (6) or λ' in (7).

The Two-Body Problem

Any two configurations of $N = 2$ particles in $\mathbb{R}^d$ are equivalent. Moreover, (7) reduces to just one equation:

$S_{12}(q_1 - q_2) = 0 \quad S_{12} = \frac{1}{r_{12}^3} - \lambda'$

which holds for $\lambda' = r_{12}^{-3}$. Thus every configuration of two bodies is central.

In this case, the possible motions are well-known – each mass moves on a conic section according to Kepler's laws. In particular, one has a family of elliptical periodic motions ranging from circular (eccentricity $\epsilon = 0$) to collinear (limit as $\epsilon \to 1$). The latter is an example of a total collision solution, that is, one for which all N bodies collide at the center of mass.

Symmetrical Configurations of Equal Masses

When all of the masses are equal, it is obvious from symmetry that certain configurations are central. In the plane ($d = 2$), one can place the masses at the vertices of a regular N-gon or at the vertices of a regular $(N-1)$-gon with the last mass at the center (see Figure 2). Similarly, in space ($d = 3$), a regular polyhedron or centered regular polyhedron (with added mass at the center) are central configurations but these are possible only for special values of N. For $d > 3$, regular polytopes are central configurations for certain values of N. Note however, that equal masses may admit other central configurations with less symmetry or with no symmetry at all.

When some but not all of the masses are equal it is again possible to look for central configurations which are symmetric under similarities of $\mathbb{R}^d$ which permute equal masses. For example, it is clear that the centered regular polygon is still a central configuration when the mass at the center is different from the others.

Euler

The first nontrivial examples of central configurations were discovered by Euler in 1767, who studied the case $N = 3$, $d = 1$, that is, three bodies on a line (Euler (1767)). When two masses are equal, one can get a central configuration by putting an arbitrary mass at their midpoint (a centered 2-gon). For three unequal masses it is not obvious that any central configurations exist. But Euler showed that, in fact, there will be exactly one equivalence class of collinear central configurations for each possible ordering of the masses along the line.

Lagrange

Lagrange found next example in the planar three-body problem $N = 3, d = 2$. Remarkably, an equilateral triangle is a central configuration, not only for equal masses, but for any three masses m_1, m_2, m_3. Moreover, it is the only noncollinear central configuration for the three-body problem (Lagrange (1772)).

When the masses are not equal, the center of mass will not be the center of the triangle and it is not at all obvious that the configuration is central. But it is easy to see it using mutual distance coordinates. The three mutual distances r_{12}, r_{31}, r_{23} can be used as local coordinates on the space of noncollinear configurations of three bodies in the plane up to symmetry. The potential and the moment of inertia can be expressed in these coordinates as

$U = \frac{m_1 m_2}{r_{12}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_3}{r_{23}} \qquad I = \frac{m_1 m_2 r_{12}^2 + m_3 m_1 r_{31}^2 + m_2 m_3 r_{23}^2}{m_1 + m_2 + m_3}.$

Now use the characterization of CC's as critical points of U with fixed I. Setting

$\frac{\partial U}{r_{ij}} + \frac{\lambda}{2}\frac{\partial I}{r_{ij}}$ gives $r_{ij}^3 = \frac{m_1+m_2+m_3}{\lambda}$, which holds for some λ if and only if the three distances are equal.

This result can be generalized to higher dimensions: the regular simplex is a CC of N bodies in $N-1$ dimensions for all choices of the masses and is the only CC spanning the full $N-1$ dimensions.

Properties of Central Configurations

- Released from rest, a CC maintains the same shape as it heads toward total collision (homothetic motion).
- Given the correct initial velocities, a CC will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit (elliptic, hyperbolic, parabolic, ejection-collision) can be attached to a CC to obtain a solution to the full nBP.
- For any collision orbit in the nBP, the colliding bodies asymptotically approach a CC
- Bifurcations in the topology of the integral manifolds (holding hc^2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to CCs.

Examples of CCs:

- Equilateral Triangle (Lagrange 1772)
- 3-Body Collinear Configuration (Euler 1767)
- Regular n-gon (equal mass required for $n \geq 4$)
- 1 + n-gon (arbitrary central mass) (Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings)

Lagrangian point wikipedia.org/wiki/Lagrangian_point

The points near two large bodies in orbit where a smaller object will maintain its position relative to the large orbiting bodies. At other locations, a small object would go into its own orbit around one of the large bodies, but at the Lagrangian points the gravitational forces of the two large bodies, the centripetal force of orbital motion, and (for certain points) the Coriolis acceleration all match up in a way that cause the small object to maintain a stable or nearly stable position relative to the large bodies.

There are five such points, labeled L1 to L5, all in the orbital plane of the two large bodies, for each given combination of two orbital bodies. For instance, there are five Lagrangian points L1 to L5 for the Sun–Earth system, and in a similar way there are five different Lagrangian points for the Earth–Moon system. L1, L2, and L3 are on the line through the centers of the two large bodies. L4 and L5 each form an equilateral triangle with the centers of the large bodies. L4 and L5 are stable, which implies that objects can orbit around them in a rotating coordinate system tied to the two large bodies.

Several planets have trojan satellites near their L4 and L5 points with respect to the Sun. Jupiter has more than a million of these trojans. Artificial satellites have been placed at L1 and L2 with respect to the Sun and Earth, and with respect to the Earth and the Moon. The Lagrangian points have been proposed for uses in space exploration.

History

The three collinear Lagrange points (L1, L2, L3) were discovered by Leonhard Euler a few years before Joseph-Louis Lagrange discovered the remaining two.

In 1772, Lagrange published an "Essay on the three-body problem". In the first chapter he considered the general three-body problem. From that, in the second chapter, he demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits.

L1 point

The L1 point lies on the line defined by the two large masses M1 and M2, and between them. It is the most intuitively understood of the Lagrangian points: the one where the gravitational attraction of M2 partially cancels M1's gravitational attraction.

Explanation

An object that orbits the Sun more closely than Earth would normally have a shorter orbital period than Earth, but that ignores the effect of Earth's own gravitational pull. If the object is directly between Earth and the Sun, then Earth's gravity counteracts some of the Sun's pull on the object, and therefore increases the orbital period of the object. The closer to Earth the object is, the greater this effect is. At the L1 point, the orbital period of the object becomes exactly equal to Earth's orbital period. L1 is about 1.5 million kilometers from Earth, or 0.01 au, 1/100th the distance to the Sun.

L2 point

The L2 point lies on the line through the two large masses, beyond the smaller of the two. Here, the gravitational forces of the two large masses balance the centrifugal effect on a body at L2.

Explanation

On the opposite side of Earth from the Sun, the orbital period of an object would normally be greater than that of Earth. The extra pull of Earth's gravity decreases the orbital period of the object, and at the L2 point that orbital period becomes equal to Earth's. Like L1, L2 is about 1.5 million kilometers or 0.01 au from Earth.

L3 point

The L3 point lies on the line defined by the two large masses, beyond the larger of the two.

Explanation

Within the Sun–Earth system, the L3 point exists on the opposite side of the Sun, a little outside Earth's orbit and slightly further from the Sun than Earth is. This placement occurs because the Sun is also affected by Earth's gravity and so orbits around the two bodies' barycenter, which is well inside the body of the Sun. At the L3 point, the combined pull of Earth and Sun cause the object to orbit with the same period as Earth.

L4 and L5 points

The L4 and L5 points lie at the third corners of the two equilateral triangles in the plane of orbit whose common base is the line between the centers of the two masses, such that the point lies behind (L5) or ahead (L4) of the smaller mass with regard to its orbit around the larger mass.

The triangular points (L4 and L5) are stable equilibria, provided that the ratio of $\frac{M1}{M2}$ is greater than 24.96. This is the case for the Sun–Earth system, the Sun–Jupiter system, and, by a smaller margin, the Earth–Moon system. When a body at these points is perturbed, it moves away from the point, but the factor opposite of that which is increased or decreased by the perturbation (either gravity or angular momentum-induced speed) will also increase or decrease, bending the object's path into a stable, kidney bean-shaped orbit around the point (as seen in the corotating frame of reference).

In contrast to L4 and L5, where stable equilibrium exists, the points L1, L2, and L3 are positions of unstable equilibrium. Any object orbiting at L1, L2, or L3 will tend to fall out of orbit; it is therefore rare to find natural objects there, and spacecraft inhabiting these areas must employ station keeping in order to maintain their position.

https://medium.com/starts-with-a-bang/throwback-thursday-earth-s-second-moon-f9bb437ac31f

Mathematical details

Lagrangian points are the constant-pattern solutions of the restricted three-body problem. For example, given two massive bodies in orbits around their common barycenter, there are five positions in space where a third body, of comparatively negligible mass, could be placed so as to maintain its position relative to the two massive bodies. As seen in a rotating reference frame that matches the angular velocity of the two co-orbiting bodies, the gravitational fields of two massive bodies combined providing the centripetal force at the Lagrangian points, allowing the smaller third body to be relatively stationary with respect to the first two.

L1

The location of L1 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} = \frac{M_2}{r^2} + \frac{M_1}{R^2} - \frac{r(M_1+M_2)}{R^3}$, where r is the distance of the L1 point from the smaller object, R is the distance between the two main objects, and M1 and M2 are the masses of the large and small object, respectively. Solving this for r involves solving a quintic function, but if the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then L1 and L2 are at approximately equal distances r from the smaller object, equal to the radius of the Hill sphere, given by: $r \approx R\sqrt[3]{\frac{M_2}{3M_1}}$. This distance can be described as being such that the orbital period, corresponding to a circular orbit with this distance as radius around M2 in the absence of M1, is that of M2 around M1, divided by $\sqrt{3} \approx 1.73$: $T_{s,M_2}(r) = \frac{T_{M_2,M_1}(R)}{\sqrt{3}}$.

L2

The location of L2 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} + \frac{M_2}{r^2} = \frac{M_1}{R^2} + \frac{r(M_1+M_2)}{R^3}$, with parameters defined as for the L1 case. Again, if the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then L2 is at approximately the radius of the Hill sphere, given by: $r \approx R\sqrt[3]{\frac{M_2}{3M_1}}$.

L3

The location of L3 is the solution to the following equation, gravitation providing the centripetal force: $\frac{M_1}{(R-r)^2} + \frac{M_2}{(2R-r)^2} = \frac{M_1}{R^2} + \frac{M_1+M_2}{R^2} - \frac{r(M_1+M_2)}{R^3}$ where r now indicates the distance of L3 from the "antipodal position" of the smaller object. If the mass of the smaller object (M2) is much smaller than the mass of the larger object (M1) then: $r \approx R + R\frac{5M_2}{12M_1}$.

L4 and L5

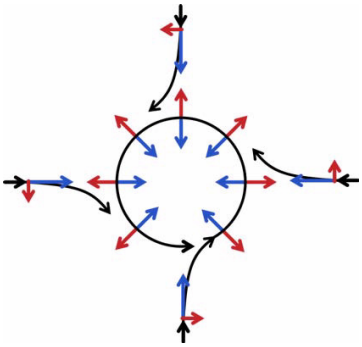
The reason these points are in balance is that, at L4 and L5, the distances to the two masses are equal. Accordingly, the gravitational forces from the two massive bodies are in the same ratio as the masses of the two bodies, and so the resultant force acts through the barycenter of the system; additionally, the geometry of the triangle ensures that the resultant acceleration is to the distance from the barycenter in the same ratio as for the two massive bodies. The barycenter being both the center of mass and center of rotation of the three-body system, this resultant force is exactly that required to keep the smaller body at the Lagrange point in orbital equilibrium with the other two larger bodies of system. (Indeed, the third body need not have negligible mass.) The general triangular configuration was discovered by Lagrange in work on the three-body problem.

Stability

Although the L1, L2, and L3 points are nominally unstable, there are (unstable) periodic orbits called "halo" orbits around these points in a three-body system. A full n-body dynamical system such as the Solar System does not contain these periodic orbits, but does contain quasi-periodic (i.e. bounded but not precisely repeating) orbits following Lissajous-curve trajectories. These quasi-periodic Lissajous orbits are what most of Lagrangian-point space missions have used until now. Although they are not perfectly stable, a modest effort of station keeping keeps a spacecraft in a desired Lissajous orbit for a long time. Also, for Sun–Earth-L1 missions, it is preferable for the spacecraft to be in a large-amplitude (100,000–200,000 km or 62,000–124,000 mi) Lissajous orbit around L1 than to stay at L1, because the line between Sun and Earth has increased solar interference on Earth–spacecraft communications. Similarly, a large-amplitude Lissajous orbit around L2 keeps a probe out of Earth's shadow and therefore ensures continuous illumination of its solar panels.

The L4 and L5 points are stable provided that the mass of the primary body (e.g. the Earth) is at least 25 times the mass of the secondary body (e.g. the Moon). The Earth is over 81 times the mass of the Moon (the Moon is 1.23% of the mass of the Earth). Although the L4 and L5 points are found at the top of a "hill", as in the effective potential contour plot above, they are nonetheless stable. The reason for the stability is a second-order effect: as a body moves away from the exact Lagrange position, Coriolis acceleration (which depends on the velocity of an orbiting object

and cannot be modeled as a contour map) curves the trajectory into a path around (rather than away from) the point.



In the *Principia* the problem of two attracting bodies with an inverse square law of force is completely solved (in Propositions 1-17, 57-60 in Book I). **Newton** argues that an inverse square law must produce elliptical, parabolic or hyperbolic orbits.

It is important at this stage to examine the problems which now arose. **Newton** had completely solved the theoretical problem of the motion of two point masses under an inverse square law of attraction. For more than two point masses only approximations to the motion of the bodies could be found and this line of research led to a large effort by mathematicians to develop methods to attack this three body problem. However, the problem of the actual motion of the planets and moons in the solar system was highly complicated by other considerations.

Even if the Earth - Moon system were considered as a two body problem, theoretically solved in the Principia, the orbits would not be simple ellipses. Neither the Earth nor the Moon is a perfect sphere so does not behave as a point mass. This was to lead to the development of mechanics of rigid bodies, but even this would not give a completely accurate picture of the two body problem since tidal forces mean that neither the Earth nor Moon is rigid.

Euler, from **1760** onwards, seems to be the first to study the general problem of three bodies under mutual gravitation (rather than looking at bodies in the solar system) although at first he only considered the restricted three body problem when one of the bodies has negligible mass. When one body has negligible mass it is assumed that the motions of the other two can be solved as a two body problem, the body of negligible mass having no effect on the other two. Then the problem is to determine the motion of the third body attracted to the other two bodies which orbit each other. Even in this form the problem does not lead to exact solutions. **Euler**, however, found a particular solution with all three bodies in a straight line.

The Académie des Sciences Prize of **1772** for work on the orbit of the Moon was jointly won by **Lagrange** and **Euler**. **Lagrange** submitted *Essai sur le problème des trois corps* in which he showed that **Euler**'s restricted three body solution held for the general three body problem. It was the first complete reduction of symetries in this problem. He discovering the so-called homographic motions, he shows that these motions necessarily take place in a fixed plane, a result which is simple only for the relative equilibria. Euler had treated only the colinear case. **Lagrange** also found another solution where the three bodies were at the vertices of an equilateral triangle. **Lagrange** believed his solutions did not apply to the solar system, but we now know the both the Earth and Jupiter have asteroids sharing their orbits in the equilateral triangle solution configuration discovered by **Lagrange**. For Jupiter these bodies are called Trojan planets, the first to be discovered being Achilles in **1908**. The Trojan planets move 60° in front and 60° behind Jupiter at what are now called the Lagrangian points.

Work on the general three body problem during the 19th Century had begun to take two distinct lines. One was the developing of highly complicated methods of approximating the motions of the bodies. The other line was to produce a sophisticated theory to transform and integrate the equations of motion. The first of these lines was **celestial mechanics** while the second was **rational** or **analytic mechanics**. Both the theory of perturbations and the theory of variations of the arbitrary constants were of major mathematical significance as well as contributing greatly to the understanding of planetary orbits.

"**Poincaré** offered the first example of chaotic behavior in a deterministic system."

“At the time of **Poincaré**, the method of solving systems of differential equations by finding first intervals was much in use. It had been known for a long time that the n-body problem had 10 independent algebraic first integrals: 3 for the center of mass, 3 for the linear momentum, 3 for the angular momentum, and one for the energy. This allowed the reduction of the primitive system from **6n** variables to $6n - 10$."

“**Jacobi** had shown that using a so-called reduction of nodes (some symmetries), the dimension of the system could be further reduced to $6n - 12$. "

"A special form of the general three body problem was proposed by **Euler** [**1767**]. He considered three bodies of arbitrary (finite) masses and placed them along a straight line. **Euler** showed that the bodies would always stay on this line for suitable initial conditions, and that the line would rotate about the center of mass of the bodies, leading to periodic motions of all three bodies on ellipses."

“Around the same time, **Lagrange** [**1772**] found a second class of periodic orbits in the general three body problem. He showed that one could position the bodies to form a triangle of equal sides in such a way that they would move along ellipses for certain initial conditions, preserving always their original configuration."

These are referred to as particular solutions of the general three body problem.

Euler [**1776**] was the first to formulate the **CR3BP** in a rotating (or synodic) coordinate system (a synodic day is the period it takes for a planet to rotate once in relation to the star it is orbiting (its primary body). For Earth, the synodic day is known as a solar day, and its mean length is 24 hours and 2.5 ms).

Subsequently, **Lagrange** demonstrated that "there were five equilibrium points (Lagrange points) at which the gravitational forces of the bodies balanced out (the Trojan asteroids discovered in **1906** along Jupiter’s orbit occupy space close to two of the Lagrange points.)"

Simplest n-body problem is when there is just one body.

Moeckel: When $n = 1$, there is no gravitational interaction, and we are reduced to the well-studied case of a free rigid body. In this case, the relative equilibrium motions are the steady rotations around the principal axes of inertia and the minimum energy motions are rotations around the **axis of maximum moment-of-inertia**.

RE of Full 2BP

Moeckel: "The 2BP is much more complicated and there is a substantial literature about it. Lagrange studied the librations of the moon in its motion around the earth [9]. If we think of the earth as spherical and unaffected by the moon, then we have the problem of a rigid body moving in a **central force field** and a phase locked moon would be a relative equilibrium even though the earth is not phase locked. Using some approximations to the potential, Lagrange found 24 relative equilibria in the generic case of a body with three distinct principle moments of inertia. These are such that the three principle axes are along the radius, tangent and normal to the orbit. The 4 minimum energy solutions have the principle axis of maximal moment of inertia along the normal and the axis of minimal moment of inertia along the radius."

Definition 2 (*Energetic Stability*) A given relative equilibrium is said to be “Energetically Stable” if any equi-energy deviation from that relative equilibrium requires a negative internal kinetic energy, $T_r < 0$, meaning that this motion is not allowed.

Note that energetic stability is different than Lyapunov or spectral stability, which are the usual notions of stability in astrodynamics (these distinctions are discussed in detail for the Full Body Problem in [8]). Energetic stability is stronger in general, as it is robust to any energy dissipation and in fact—if it applies—means that a given relative equilibrium configuration cannot shed any additional energy and thus is static without the injection of exogenous energy, a condition we refer to as being in a minimum energy state.

Maciejewski 1995 - "The change from a material point to a rigid body manifests itself not only in larger number of equations of motion, but also in a much more complicated form of the right hand sides of equations of motion. Even if all, except one, bodies are points, the right hand sides depend on infinitely many parameters. These parameters themselves depend on the mass distribution and the shape of the body."

"Using the obtained reduced equations, we prove in section 4 the existence of relative equilibria, and we show that, in general, they are not Lagrangean, i.e., bodies move in circular orbits, but planes of these orbits do not coincide."

Scheeres 2006 - "The study of relative equilibria between massive bodies has received much interest over the years as many solar system bodies have been found to lie in or close to relative equilibria. Examples include the moon relative to the Earth, the Galilean satellites relative to Jupiter, and more recently asteroid binary secondaries with respect to their primaries."

"The study of this problem requires one to solve for and analyze the mutual gravitational attractions and torques between two non-spherical bodies."

"Usually, one of the bodies (generally the larger) is not in a synchronous rotation state, giving rise to periodic perturbations to the system."

"Sphere-Restricted Full 2-Body Problem (SRF2BP), which arises when one of the bodies is assumed to be a sphere (i.e., a point mass)."

"We suspect that most relative equilibria between two general asteroids will be unstable when the distance is close, and only become stable as their distance increases, consistent with ellipsoidal models"

Scheeres 2009 - "Stability of the Full Two-Body Problem is studied in the case where both bodies are non-spherical, but are restricted to planar motion." "We identify all relative equilibria and determine their stability properties, with an emphasis on finding the energetically stable relative equilibria and conditions for Hill stability of the system." "Classical studies,..., have often focused on applying perturbation theory to the problem, under the assumption that one or both of the bodies are nearly spherical and that the coupling between rotational and translational motion is weak and may even be separated from each other." "Any contact binary that has been spun to fission will initially lie in an unstable relative equilibrium. A study of the properties of this equilibrium state show that, in the absence of energy dissipation, the system may undergo disruption (i.e., mutual escape) or be trapped in a chaotic dynamical state as a function of the body moments of inertia and the mass fraction between the bodies."

"Granular Systems" "Finite Density Distributions" "Extended Rigid Bodies"

"All binary systems that arise from the fission of the two components will be unstable, with a stable relative equilibrium in existence at a larger distance with a lower system energy (but same system angular momentum). This has implications for the evolution of an asteroid spun to fission, implying that it immediately enters a dynamic phase which can include reimpact of the bodies or separation of the bodies. If the total system energy is positive, the system may even disrupt. A surprising find is that these stability considerations also apply to a system of two spheres orbiting each other, when their rotational moments of inertia are incorporated into the energy and angular momentum constraints. This shows that not every circular orbit of a Keplerian system is stable, and that as a function of the rotational moments of inertia a different circular orbit may be found at the same value of angular momentum which has a lower value of system energy. A second important result is that the stability constraints and equilibrium separations between two strongly non-spherical bodies, when scaled appropriately, is close to the same results as stated for a two-sphere system."

"This problem has been shown to be relevant to the understanding of Solar System bodies, especially among asteroids whose size is small enough so that when their components rest on each other they have insufficient gravitational attraction to overcome material strength, and thus retain the physical characteristics of rigid bodies resting on each other."

Scheeres 2012 - "Celestial Mechanics systems have two fundamental conservation principles that enable their deeper analysis: conservation of momentum and conservation of (mechanical) energy." "real systems always have a finite density and, hence, any particle in a celestial mechanics system has a finite size. Such a physically corrected system has been called the “Full N-Body Problem,” as inclusion of finite density also necessitates the modeling of the rotational motion of the components, which is not needed for consideration of point masses. It also necessitates

consideration of contact forces as their mass centers cannot come arbitrarily close to each other, as at some distance they will rest on each other."

"In Scheeres 2012 the Full N-body problem is introduced, distinguished from the traditional N-body problem in that each body has a finite density, and hence two bodies cannot come arbitrarily close to each other. The finite-sized bodies are assumed to be rigid, and only exhibit contact forces between each other of friction and coefficient of restitution. This means that relative equilibria must also include resting configurations, with all relative motion between components zeroed. This both results in new possible relative equilibrium configurations of an N-body system, with components resting on each other, and also places lower limits on the gravitational potential energy. This model has been developed to describe the relative mechanics and dynamics of self-gravitating rubble pile asteroids, where the relative forces between components are weak enough for the rigidity of the components to not be compromised. In Scheeres 2012 it is shown how this simple change provides a drastic modification of the stable states of the 2 and 3 body problems. For the 3-body problem under the restriction that the bodies are equal mass spheres these relative equilibria are shown in Fig. 1, and in addition to the classical Euler and Lagrange solutions also include the so-called, Resting Lagrange, Transitional, Transverse and Aligned.

Hill Stability in the Full 3-Body Problem

Equilibria in the Spherical, Full 3 Body Problem with Equal Size

Configuration	Name	Energetic Stability	Conditions
	Lagrange Resting	Stable	
	Euler Resting	Stable	For high enough H
	Aligned	Stable	Outer Solution
	Lagrange	Unstable	
	Euler	Unstable	
	Euler Resting	Unstable	For low H
	Aligned	Unstable	Inner Solution
	Transverse	Unstable	
	Transitional	Unstable	

Figure 1. Relative equilibria in the equal mass, finite density 3-body problem.

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It is important to note that the energy and angular momentum integrals contain terms related to both rotation and translation, meaning that energy and momentum can be transferred from one mode to another during interactions. For nonspherical bodies in close proximity to each other (a few radii) it is possible for very large transfers of energy and momentum to be made. The mechanisms of these transfers are discussed in more detail in Scheeres et al. (2000) and Scheeres (2001a).

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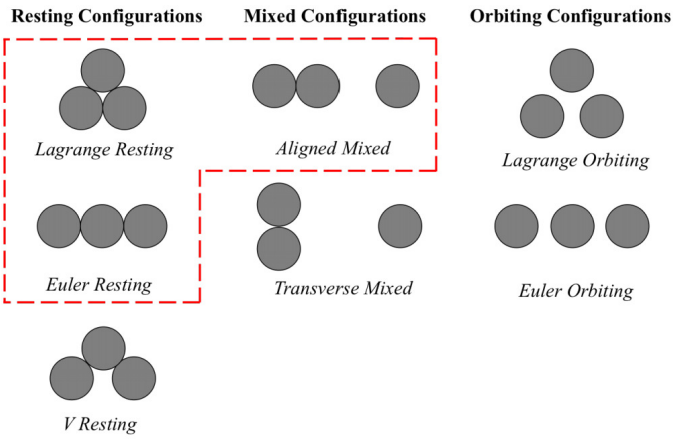


3.1 No Contact Case

When there are no contacts between the bodies, there are necessarily no active unilateral constraints and all of the degrees of freedom are unconstrained. We also note that the kinetic energy is quadratic in the generalized coordinate rates and has the form of a natural system ([2], p. 72). Then the condition for a relative equilibrium are that all of the $\dot{\mathbf{q}} = 0$ (yielding $T_r = 0$) and $\partial \mathscr{E} / \partial \mathbf{q} = \mathbf{0}$. Energetic stability of the configuration occurs when the Hessian of the amended potential is positive definite, or $\partial^2 \mathscr{E} / \partial \mathbf{q}^2 > 0$, meaning that it has only positive eigenvalues. Neutral stability can occur when $\partial^2 \mathscr{E} / \partial \mathbf{q}^2 \geq 0$, meaning that at least one eigenvalue is equal to zero. In this case it is possible for the system to drift at a constant rate relative to the equilibrium, ultimately destroying the configuration. If the configuration is not positive definite or semi-definite, then there exists at least one negative eigenvalue and the system can escape from the equilibrium configuration while conserving energy. Another way to consider this case is that the system can still dissipate energy, and thus can evolve to a lower energy state. We note that this is a stronger form of stability than is sometimes used in celestial mechanics and astrodynamics, where spectral stability of linearized motion can sometimes be stable (as in the Lagrange configurations of the 3-body problem that satisfy the Routh criterion)

- 3-Body Collinear Configuration (Euler 1767)

"He considered three bodies of arbitrary (finite) masses and placed them along a straight line. Euler showed that the bodies would always stay on this line for suitable initial conditions, and that the line would rotate about the center of mass of the bodies, leading to periodic motions of all three bodies along ellipses."

"As a special case, the circular Keplerian orbits give rise to a homographic solution for which the configuration just rotates rigidly at constant angular speed. This is called a relative equilibrium solution and planar CC's are often called relative equilbria (RE's). These are true equilibrium solutions in a uniformly rotating coordinate system or in a rotation-reduced system where the rotational symmetry of the N-body problem has been eliminated.

"The three collinear Lagrange points (L1, L2, L3) were discovered by Leonhard Euler a few years before Joseph-Louis Lagrange discovered the remaining two. In 1772, Lagrange published an "Essay on the three-body problem". In the first chapter he considered the general three-body problem. From that, in the second chapter, he demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits."

"He showed that if the bodies were positioned in such a way that they form a triangle of equal sides they would move along ellipses for certain initial conditions, preserving always their original configuration."

When does Energetic Stability Occur? When the Hessian of the amended potential is positive definite or $\partial_q^2 U_{red}$ has only positive eigenvalues. With one negative eigenvalue, the system can escape from RE while conserving energy.

When does the Hamiltonian represent the energy constant of motion? If the Lagrangian (and therefore Hamiltonian) is not an explicity function of time. Often this is not the case in rotating reference frames.

Turning 2BP eq: $m\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right) = 0$ **into a constant of motion**:
$m\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right) = \frac{m}{r}\left(r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi}\right) = \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\varphi}\right) = 0$, or $r^2\dot{\varphi} =$ constant $= h$.

Way to show we have Conserved Total Energy w/NBP? CC $\Rightarrow$ Conservative Force $\Rightarrow$ Conserved Total Energy.

Recall in 2BP: $L = mr^2\dot{\varphi}$. **So**, $r^2\dot{\varphi} = h$, **const**. **If** $h \neq 0$, **then**: Curvilinear Sector of area swept out by $\vec{r}$ is: $S(t) = \frac{ht}{2}$, thus $\dot{S} = \frac{h}{2}$ and the sector velocity is constant. “area integral" or “Kepler’s 2nd Law". h is “area constant."

When is a force called conservative? If there’s a potential V such that the components of force can be written as $F_i = -\frac{\partial V}{\partial x_i} \equiv -\partial_i V$. Gravity and electrostatic force satisfy this.

Pros of Lagrange Eqs vs Newton’s Laws of Motion: Lagrange’s EQ hold in arbitrary curvilinear coordinate system. # of Lagrange EQs= # of degrees of freedom. Newton: 3 EQs for each body & possibly constraint EQs.

Derive Newton’s Force Law from Lagrange’s Equations: Euler-Lagrange EQ: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$. Observe: $\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i = p_i$. So, $\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial x_i}$. Observe: $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial}{\partial x_i}(T - V) = -\frac{\partial V}{\partial x_i}$, since T does not depend on x_i. Observe: $F_i := -\frac{\partial V}{\partial x_i}$, therefore $\frac{dp_i}{dt} = F_i$, Newton’s Law of Force.

Generalized Momentum conjugate to q_i for Hamiltonian: Defined to be: $p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$.

Legendre Transformation of $f(x_1, \ldots, x_n) \equiv f(x)$. Let: $y_i = \frac{\partial f}{\partial x_i}$, then $\Sigma x_i y_i - f$ is the Legendre Transformation. Hamiltonian $\mathcal{H} = \Sigma p_i \dot{q}_i - \mathcal{L}$ is a transformation of $\mathcal{L}$.

Differences between Hamilton’s Eqs & **Lagrange’s Eqs**: Both hold in arbitrary curvilinear coordinate systems. Both EOM derived from scalar functions $\mathcal{L}$ or $\mathcal{H}$. $\mathcal{H}$: 1st-order, $\mathcal{L}$: 2nd-order. $\mathcal{L}$: One EQ per degree freedom. $\mathcal{H}$: Two EQ per degree freedom; one for q_i, and one for p_i.

How to Integrate Hamiltonian Problem: For an integral F, sols lie on $F^{-1}(c)$ with dim $= 2n - 1$. If you have $2n$–1 such independent ($\{F_i, F_j\} = \Sigma_s\left(\frac{\partial F_i}{\partial q_s}\frac{\partial F_j}{\partial p_s} - \frac{\partial F_i}{\partial p_s}\frac{\partial F_j}{\partial q_s}\right) = 0,\ \forall i \neq j$) integrals F_i, then holding these fixed would define a !sol curve in $\mathbb{R}^{2n}$.

Hamiltonian Stable RE: An equilibrium point z_0 is stable if for every $\varepsilon > 0$, $\exists \delta > 0$ such that $|z_0 - \varphi(t, z_1)| < \varepsilon$, $\forall t$ whenever $|z_0 - z_1| < \delta$.

Define: $V: O \to \mathbb{R}$ **pos**. **def**. **wrt f.p**. z_0 **of** $\dot{z} = f(z)$ **smooth**: If there is a neighborhood $Q \subset O$ of z_0 such that $V(z_0) < V(z),\ \forall z \in O \backslash \{z_0\}$. And z_0 is called a strict local minimum of V.

Lyapunov's Stability Theorem: If there exists a function V that is positive definite wrt z_0 and such that $\dot{V} \leq 0$ in a neighborhood of z_0, then the equilibrium z_0 is positively stable (as $t \to \infty$).

Dirichlet's Stability Theorem: If z_0 is a strict local minimum or maximum of H, then z_0 is a stable equilibrium of $\dot{z} = J\nabla H(z)$.

Chetaev's Thm for $\dot{z} = J\nabla H(z) = f(z)$: $V : O \to \mathbb{R}$ a smooth function & Ω an open subset of O w/: $z_0 \in \partial\Omega$. Also: $V > 0$ for $z \in \Omega$. $V = 0$ for $z \in \partial\Omega$. $\dot{V} = V \cdot f > 0$ for $z \in \Omega$. Then, f.p. z_0 is unstable. $\exists N(z_0)$ such that sols in $N \cap \Omega$ leave N in positive time.

Requirement for Generalized Coordinates: Span the space of the motion in phase space, and be linearly independent. Often found by: $p_i := \partial_{\dot{q}_i} \mathcal{L}$.

Requirements for Solving 2BP: 2 Integrals of Motion (L, H) and two initial values (φ_0, r_0).

Poission Bracket of F and G : $\{F, G\} = \Sigma_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$. Skew-symmetric and bilinear. In terms of phase variable $\vec{z}$, bracket of $F(\vec{z}, t), G(\vec{z}, t)$ is $\nabla F \cdot J \nabla G$. Associated with every Hamiltonian H is a vector field defined by: $\hat{v}_H(F) = \{F, H\}$.

Cyclic Hamiltonian Coordinate φ : Does not appear in Hamiltonian. Momentum ($p = m\,\dot{\varphi}$) conjugate to φ is integral of motion. Associated w/symmetry of system. Noether identified this correspondence. Generalized momentum $p = \frac{\partial L}{\partial \dot{q}}$, from Euler Lagrange $\frac{d}{dt} \partial_{\dot{q}} L = \partial_q L = 0$. So, p is conserved.

NBP. **How force on each mass $\vec{f}_i$ is derived given**: $\mathbf{V}\left(\vec{r}\right) = -\mathbf{\Sigma}_{i<j} \frac{G m_i m_j}{|\vec{r}_j - \vec{r}_i|}$. $\vec{f}_i = -\nabla_{\vec{r}_i} \mathbf{V}\left(\vec{r}\right) = \Sigma_{j \neq i} \vec{f}_{ij}\left(\vec{r}\right)$ where $\vec{f}_{ij} = \frac{G m_i m_j}{|\vec{r}_j - \vec{r}_i|^3} \left(\vec{r}_j - \vec{r}_i\right)$ and $\vec{r} = \left(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n\right)$.

Potential for ERB: $dm_i = \rho_i\left(\vec{a}_i\right) d\vec{a}_i$, where $\vec{a}_i$ position in body frame $\mathcal{F}_i$ for $\mathcal{B}_i$ and ρ_i density distribution. $V_{ij}\left(\vec{\theta}, \vec{r}\right) = -G \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{1}{|(\vec{r}_i - \vec{r}_j) + \mathbf{B}_i \vec{a}_i - \mathbf{B}_j \vec{a}_j|} dm_j dm_i$, where $\mathbf{B}_i\left(\vec{\theta}_i\right)$ is the transformation matrix in Euler angles $\vec{\theta}_i = (\psi_i, \theta_i, \varphi_i)$ from body frame to inertial frame. So, $V\left(\vec{r}, \vec{\theta}\right) = \Sigma_{i<j} V_{ij}$.

"**Natural**" **Hamiltonian**: $H(q,p) = T(q,p) + U(q)$, where T is the kinetic energy, and U the potential energy. No time dependence.

Variational Equation for Equilibrium z_0 : Assume $\delta z = z - z_0$ is infinitesimal, Variational Eq is $\delta\,\dot{z} = L\delta z$, where constant matrix $L = JD^2H(z_0)$ is the linearization. Solution is called the "tangent flow." Assuming distinct evals, it has the form: $\delta z = \Sigma_j c_j v_j e^{\sigma_j t}$, w/$\sigma_j$ evals & v_j evects.

"**Hamiltonian Matrix**" L: $2n \times 2n$ matrix L such that JL is symmetric, where J is the Poisson matrix, and $L^T J + JL = 0$. Example: $JD^2H(z_0) =: L$.

Eigenvalues of Hamiltonian Matrix: Come in pairs $\pm\sigma$. Therefore, Exponential growing terms exists unless all $\sigma \in i\mathbb{R}$. Thus, Linear Stability reduces to finding eigenvalues and eigenvectors of Hamiltonian Matrix L.

Lyapunov Stability of Hamiltonian Systems: Equilibrium $z_0 \in \mathbb{R}^{2n}$ is Lyapunov stable (nonlinearly stable) if for every neighborhood V of z_0, there exists a neighborhood $U \subseteq V$ such that $z(0) \in U \Rightarrow z(t) \in V, \forall t \in \mathbb{R}$.

Linear Stability of Hamiltonian Systems: F.p. $z_0 \in \mathbb{R}^{2n}$ is linearly stable if all orbits $z(t)$ of tangent flow are bounded $\forall t$. Thus, nonlinear much stronger than linear stability, as sets U & V where $z(t)$ begin don't have to be infinitesimally small. Need $\sigma \in i\mathbb{R}$ (like spectral), AND 1D Jordan blocks.

Spectral Stability of Hamiltonian Systems: Equilibrium $z_0 \in \mathbb{R}^{2n}$ is spectrally stable if $\sigma \in i\mathbb{R}$. If in addition, 1D Jordan blocks⇒Linearly stable.

Orbital Stability of Hamiltonian: Describes the divergence of two neighboring orbits, regarded as point sets.

Structural Stability of Hamiltonian: Sensitivity (or insensitivity) of the qualitative features (f.p. & invariant sets) of a flow to changes in parameters.

Hamiltonian Loss of Spectral Stability: $H(z,\mu)$ smooth in $\mu \Rightarrow \sigma$ also smooth in μ. Stability loss due to: $\sigma_{1,2} = \pm i\omega_1$ & $\sigma_{3,4} = \pm i\omega_2$ merge @0, & split onto $\mathbb{R}$ (saddle-nde). Or $\sigma_{1,2}$, $\sigma_{3,4}$ collide @$z_0, \bar{z}_0 \neq 0$ & split off into complex plane forming complex quadruplet (Krein bifurcation)

Hamiltonian Reduced Characteristic Polynomial: Since σ in ±pairs, characteristic polynomial P_{2n} is even. Introducing $\tau := -\sigma^2$ gives: $Q_n(\tau) = (-1)^n P_{2n} = \tau^n - A_1\tau^{n-1} + \ldots + (-1)^n A_n$. ⇒ Hamiltonian f.p.s are spectrally stable ⇔ all zeros of $Q_n(\tau)$ are positive.

Lagrange-Dirichlet Theorem: Let the 2nd variation of the Hamiltonian $\delta^2 H$ be definite at an equilibrium z_0. Then, z_0 is stable. $\delta^2 H := \frac{d^2}{dt^2} H(z_0 + th)|_{t=0}$

Relative Equilibrium: Solution which becomes an equilibrium in some uniformly rotating a coordinate system. F.p. of dyn sys which has been reduced through quotienting out of rotation angle. Critical points of an "amended potential"

History RE for ERBs in F2BP: Maciejewski: 36 non-Lagrangian RE as $\vec{r} \to \infty$. Scheeres: Nec/Suff for pt/ERB. Moeckel: lower bounds on # of RE for F2BP where radius of the system is large, but finite.

How to Reduce in orbital RE? For RE, invariance of orbit requires uniform rot. w/fixed $\vec{L}$ & r. So, symmetry of φ about $\vec{L}$, not found in $\mathcal{L}$. Symmetry gives first integral & allows elimination of velocity variable by solving for it explicitly in an EOM.

How solve general point mass 2BP: Change of variables such that 2BP→R2BP. $\vec{r} := \vec{r}_2 - \vec{r}_1$. $M := \frac{M_1 M_2}{M_1 + M_2}$. Then, apply sol. for Kepler Problem.

Central Force on m_i: The force on m_i is always directed toward, or away from a fixed point O; and
The magnitude of the force only depends on the distance r of m_i from O.

Central Force Motion is Planar: init pos $\vec{r}$ & vel $\vec{v}$ vectors define a plane. $\vec{r} \cdot \vec{L} = \vec{r} \cdot \left(\vec{r} \times m\vec{v}\right)$ $= m\vec{v} \cdot \left(\vec{r} \times \vec{r}\right) = 0$. $\vec{r}$ & $\frac{d\vec{r}}{dt}$ always lies in plane perpendicular to $\vec{L}$. $\vec{L}$ is constant⇒ $\vec{F}$ in plane.
$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = (\vec{v} \times m\vec{v}) + \left(\vec{r} \times m\frac{d}{dt}\vec{v}\right) = \vec{r} \times \vec{F}$. And, CF⇒ $\vec{r} \times \vec{F} = 0$.

Derive Pot. from Newton law of gravitation be**tween** m & M: $\vec{F}\left(\vec{r}\right) = -\frac{GMm}{r^3}\vec{r}$. Integrating we find: $U(r) = -\int_\infty^{\vec{r}} \vec{F}\left(\vec{s}\right) \cdot d\vec{s} = -\int_\infty^{\vec{r}} -\frac{GMm}{s^3}\vec{s} \cdot d\vec{s} = \int_\infty^r \frac{GMm}{s^2} ds = -\frac{GMm}{r}$. And Kinetic is:
$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{q}^2\right)$.

Find reduced Lagrangian $\mathcal{L}_{red}$ w/Red. EOM: $\mathcal{L}_{red} = T_{red} - U_{red}$, where T_{red} and U_{red} those necessary for $\frac{d}{dt}\frac{\partial \mathcal{L}_{red}}{\partial \dot{r}_i} - \frac{\partial \mathcal{L}_{red}}{\partial r_i}$ to produce reduced EOM.

Usefulness of conservation of linear momentum: We can assume the system's COM moves at a constant rate. This allows us to choose an inertial reference frame such that our choice of origin coincides with the system's COM. This eliminates 6 constants of motion.

Simplify Equations with ratio variables $\alpha_1 + \alpha_2 = 1$: Let $\alpha_1 = \frac{u}{1+u}$ and $\alpha_2 = \frac{1}{1+u}$. Note that we still have $\alpha_1 + \alpha_2 = 1$, but now we have characterized them with one variable $0 < u < \infty$.

Descartes' Rule of Signs for f: # of positive roots is at most the # of sign changes in sequence of f's coefficients (omitting zero coefficients), and that difference between these two #s is always even. This implies that if the # of sign changes is 0 or 1, then there are exactly 0 or 1 positive roots, resp.

Graph of $f' = f'' = 0$: Points in the space at which extremums and inflection points collide and annihilate $\backsim \rightarrow \setminus$

Conic Sections: $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$, where one of A,B,C are non-zero. All circles are similar. 2 ellipses are similar $\Leftrightarrow$ ratios of lengths of minor axes to lengths of major axes are equal.

Calculating Constants of Motion in 2BP: From the Euler Lagrange equations, we get the equations of motion: $m_1 \ddot{r}_1 + \frac{Gm_1m_2(r_1-r_2)}{d_{12}^3}$ and $m_2 \ddot{r}_2 + \frac{Gm_1m_2(r_2-r_1)}{d_{12}^3}$. If we add these equations, we get: $m_1 \ddot{r}_1 + m_2 \ddot{r}_2 = 0$, which can be integrated twice with respect to time to yield: $m_1r_1 + m_2r_2 = at + b$. Note that a and b are vectors and so contain six linearly independent constants. From the definition of the center of mass, we can write: $Mr_c = at + b$ which says that at time $t = 0$, the center of mass was located at $\frac{b}{M}$ and was moving with a uniform velocity $\frac{a}{M}$. Thus, we have found 6 of the 12 constants of motion. They are the locations of the velocity and the center of mass.

"**Integrating a Differential Equation**": The classical conserved quantities F of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where c is a constant, are invariant sets. In general, level sets are manifolds of dimension $2n - 1$, and so with an integral F, if $x_0 \in F^{-1}(c)$, then the solution lies in the set $F^{-1}(c)$, which is of dimension $2n - 1$. If you were so lucky as to find $2n - 1$ independent integrals, $F_1, \ldots, F_{2n-1}$, then holding all these integrals fixed would define a curve in $\mathbb{R}^{2n}$, the solution curve. In the classical sense, the problem has been integrated.

Sturm's Theorem

In mathematics, the Sturm sequence of a univariate polynomial P is a sequence of polynomials associated with P and its derivative by a variant of Euclid's algorithm for polynomials. Sturm's theorem expresses the number of distinct real roots of P located in an interval in terms of the number of changes of signs of the values of the Sturm sequence at the bounds of the interval. Applied to the interval of all the real numbers, it gives the total number of real roots of P.

Whereas the fundamental theorem of algebra readily yields the overall number of complex roots, counted with multiplicity, it does not provide a procedure for calculating them. Sturm's theorem counts the number of distinct real roots and locates them in intervals. By subdividing the intervals containing some roots, it can isolate the roots into arbitrary small intervals, each containing exactly one root. This yields the oldest real-root isolation algorithm, and arbitrary-precision root-finding algorithm for univariate polynomials.

The **Sturm chain** or **Sturm sequence** of a univariate polynomial $P(x)$ with real coefficients is the sequence of polynomials $P_0, P_1, \ldots$, such that $P_0 = P$, $P_1 = P'$, $P_{i+1} = -Rem(P_{i-1}, P_i)$, for $i \geq 1$, where P' is the derivative of P, and $Rem(P_{i-1}, P_i)$ is the remainder of the Euclidean division of P_{i-1} by P_i. The length of the Sturm sequence is at most the degree of P. The number of sign variations at ξ of the Sturm sequence of P is the number of sign

changes–ignoring zeros—in the sequence of real numbers $P_0(\xi), P_1(\xi), \ldots$ This number of sign variations is denoted here $V(\xi)$. Sturm's theorem states that, if P is a square-free polynomial, the number of distinct real roots of P in the half-open interval $(a, b]$ is $V(a) - V(b)$.

The theorem extends to unbounded intervals by defining the sign at $+\infty$ of a polynomial as the sign of its leading coefficient (that is, the coefficient of the term of highest degree). At $-\infty$ the sign of a polynomial is the sign of its leading coefficient for a polynomial of even degree, and the opposite sign for a polynomial of odd degree. In the case of a non-square-free polynomial, if neither a nor b is a multiple root of P, then $V(a) - V(b)$ is the number of distinct real roots of P.

Example: Suppose we wish to find the number of roots in some range for the polynomial $p(x) = x^4 + x^3 - x - 1$. So $p_0 = p$. $p_1 = p'$. The negative remainder of the Euclidean division of p_0 by p_1 is $p_2 = \frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}$. Now dividing p_1 by p_2 and multiplying the remainder by -1: $p_3 = -32x - 64$. Dividing p_2 by p_3 and multiplying the remainder by -1: $p_4 = -\frac{3}{16}$. As this is a constant, this finishes the computation of this term sequence.

To find the number of real roots of p_0, one has to evaluate the sequences of the signs of these polynomials at $-\infty$ and ∞, which are respectively $(+,-,+,+,-)$ and $(+,+,+,-,-)$. Thus, $V(-\infty) - V(\infty) = 3 - 1 = 2$, which shows that p has 2 real roots.

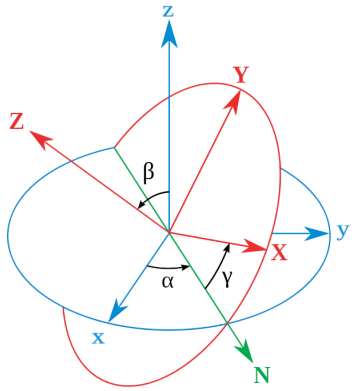
Sturm's Thm for polynml Q(τ): Sequence: $\{F_k(\tau)\}$ by $F_0(\tau) := Q(\tau)$, $F_1(\tau) := Q'(\tau)$. At each stage divide, $\frac{F_{k-2}}{F_{k-1}}$ to get G_{k-1} + Remainder = $G_{k-1} - \frac{F_k}{F_{k-1}}$, so $F_{k-2} = G_{k-1}F_{k-1} - F_k$, where $\deg F_k < \deg F_{k-1}$. $V(\tau)$:=(# of variations in sign). # of !(roots) in $(a, b]$ is $V(a) - V(b)$.

Spectral Stability via Sturm's Thm: Recall that for Hamiltonian stability zeros of reduced $Q(\tau)$ must be nonnegative real. Via Sturm's Thm, this is true $\Leftrightarrow$ $V(0) - V(\infty) = n$. For Natural systems, this implies nonlinear stability as well.

Relationship between Work, **Force**, **Potential**:
The small amount of work δW that occurs over an instant of time dt is calculated as: $\delta W = \vec{F} \cdot d\vec{s} = \vec{F} \cdot \vec{v} dt$, where the $\vec{F} \cdot \vec{v}$ is the power over the instant dt. The sum of these small amounts of work over the trajectory of the point yields the work: $W = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{s}}{dt} dt = \int_C \vec{F} \cdot d\vec{s}$.

If the work for an applied force is independent of the path, then the work done by the force, by the gradient theorem, defines a potential function which is evaluated at the start and end of the trajectory of the point of application. This means that there is a potential function $U(x)$, that can be evaluated at the two points $x(t_1)$ and $x(t_2)$ to obtain the work over any trajectory between these two points. It is tradition to define this function with a negative sign so that positive work is a reduction in the potential, that is: $W = \int_C \vec{F} \cdot d\vec{x} = \int_{\vec{x}(t_1)}^{\vec{x}(t_2)} \vec{F} \cdot d\vec{x} = U(\vec{x}(t_1)) - U(\vec{x}(t_2))$.

The function $U(x)$ is called the potential energy associated with the applied force. The force derived from such a potential function is said to be conservative. Examples of forces that have potential energies are gravity and spring forces. In this case, the gradient of work yields $\nabla W = -\nabla U = -(\partial_x U, \partial_y U, \partial_z U) = \vec{F}$, and the force F is said to be "derivable from a potential."



Euler Angles: The axes of the original frame are denoted as x, y, z and the axes of the rotated frame as X, Y, Z. The geometrical definition (sometimes referred to as static) begins by defining the line of nodes as the intersection of the planes xy and XY (it can also be defined as the common perpendicular to the axes z and Z and then written as the vector product $N = z \times Z$). Using it, the three Euler angles can be defined as follows:

- α (or φ) is the angle between the x axis and the N axis (x-convention - it could also be defined between y and N, called y-convention).
- β (or θ) is the angle between the z axis and the Z axis.
- γ (or ψ) is the angle between the N axis and the X axis (x-convention).

Square Free Polynomial: A polynomial defined over a field that does not have as a factor any square of a non-unit factor. In the important case of univariate polynomials over a field k, this means that $f \in k[X]$ is square-free if and only if $b^2 \nmid f$ for every polynomial $b \in k[X]$ of positive degree.

Why are all the bounded orbits are periodic?

Because Bertrand's theorem states that among central-force potentials with bounded orbits, there are only two types such that all bounded orbits are also closed periodic: (1) an inverse-square central force such as the gravitational or electrostatic potential: $V(r) = -\frac{k}{r}$, and (2) the radial harmonic oscillator potential: $V(r) = \frac{k}{2} r^2$.

Why can the two body collisions be "regularized".

For $\dot{y} = -\alpha |x|^{-\alpha-2}$, there is a restriction on α of $\alpha = 2(1 - \frac{1}{n})$ for $n \in \mathbb{Z}^+$, for a body to be regularizable. And for Kepler problem, $\alpha = 1$ or $n = 2$.

Originally, Sundman came up with a technique of regularizing the 2 body problem, but his technique does not guarantee smoothness of the vector flow with respect to initial data. Levi-Civita subsequently developed such a technique that focuses on the singularity of the differential equation, instead of the solutions. This technique guarantees a smoothness of the vector flow with respect to initial data, and therefore guarantees us information about flows that are close to a collision. Easton improved upon this with a more systematic approach which uses an isolating block to examine whether orbits passing close to collision determine an extension for an orbit ending in collision (block regularization). Let the 2 body problem $\dot{x} = y$ and $\dot{y} = -\alpha |x|^{-\alpha-2} x, \quad (3.2)$

for some $\alpha > 0$, with $(x(t), y(t))$ as its solution. Let $x = r^\gamma e^{i\theta}$, and $y = r^{-\beta\gamma}(v + iw)e^{i\theta}$, where $\beta = \frac{\alpha}{2}$ and $\gamma = \frac{1}{\beta+1}$. Then, (3.2) becomes:

$$\dot{r} = (\beta + 1)v, \qquad \dot{\theta} = \frac{w}{r}, \qquad \dot{w} = \frac{\beta-1}{r} wv, \qquad \dot{v} = \frac{w^2 + \beta(v^2-2)}{r}. \qquad (4.3)$$

Let: $\mathbf{M}(h) = \left\{(r,\theta,w,v) \in \mathbb{R}^4 : r \geq 0 \text{ and } (4.3) \text{ holds}\right\}$, and $\mathbf{N} = \left\{(r,\theta,w,v) \in \mathbf{M}(h) : \vec{r} = 0\right\}$.

Theorem 7.1: The singularity set **N** for system (4.3) is block regularizable if and only if $\beta = 1 - n^{-1}$, where n is a positive integer.

Note that for $n = 1$, we have $\alpha = 0$ (no force), and therefore just a free-floating mass. For $n = 2$, we have $\alpha = 1$ (the Kepler problem), and therefore it is block regularizable.