MATH 1271: Calculus I

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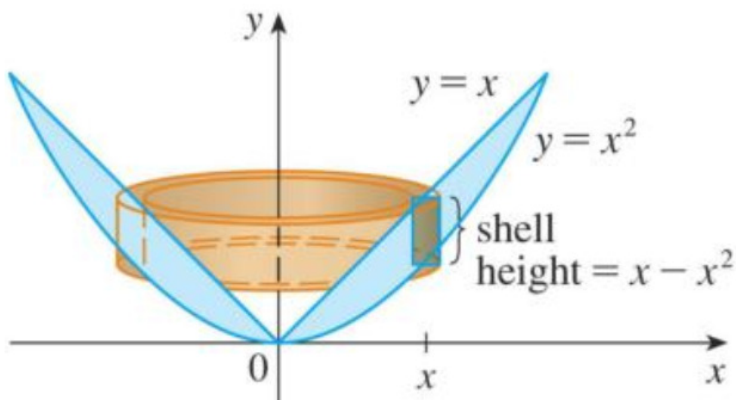
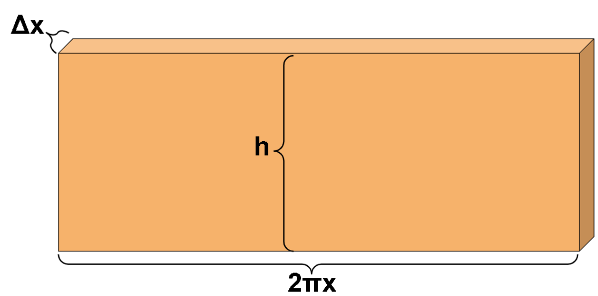
6.3 - Volumes by Cylindrical Shells

Review:



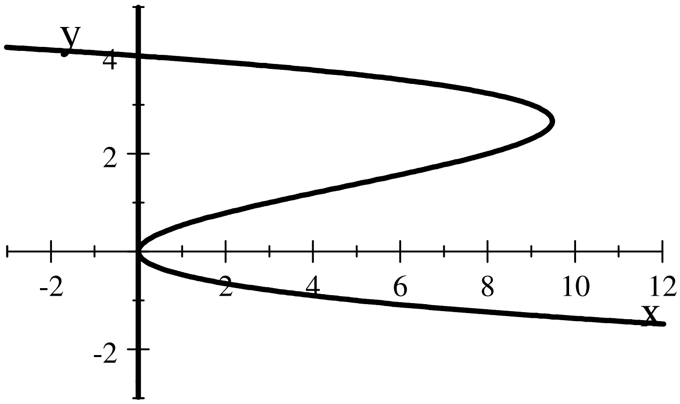


Volumes by Cylindrical Shells

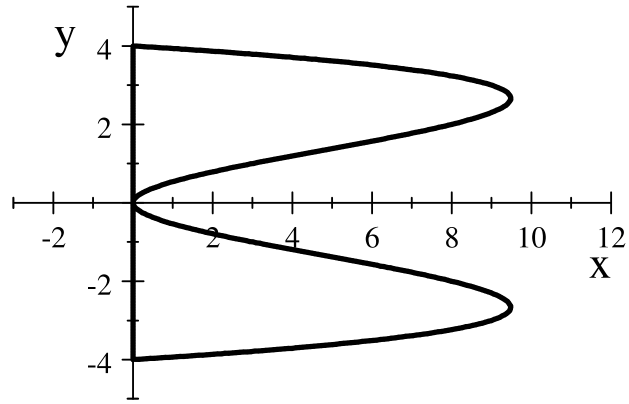
- Volume of each shell: $V = 2\pi r(x) \cdot h(x) \cdot \Delta x =$ [circumference] • [height] • [thickness]
- Exact volume of solid: $V = 2\pi \int_a^b r(x)h(x)dx,$ where $0 \leq a < b.$

What's the difference between the two methods?
The **Disk/Washer** method adds up all the cross-section areas which were **perpendicular** to the axis of rotation (areas of circles). In the method of **Cylindrical Shells**, we are adding up all the cross-section areas which are **parallel** to the axis of rotation (see the image above). Depending upon the specific problem you are trying to solve, one or the other of these methods might be more convenient to use.

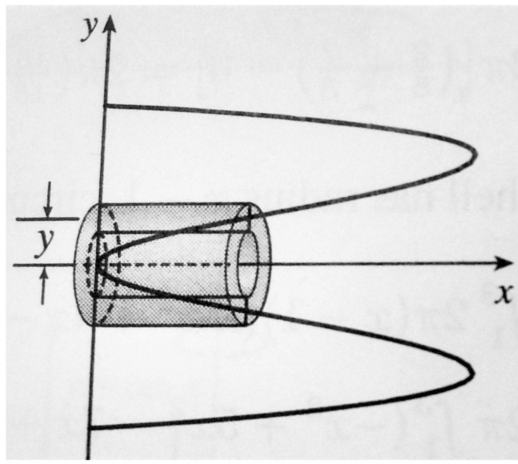
Problem 12. **Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $x = 4y^2 - y^3$ and $x = 0$ about the x-axis.**



$x = 4y^2 - y^3$, and $x = 0$



Bounded Region rotated around x-axis



What is the radius? What is the circumference?

What is the height of the cylindrical shell? bounds of Integration?

Radius: $r = y$.

Circumference: $2\pi r = 2\pi y$.

Height $h(y)$: $4y^2 - y^3$.

What are the **bounds of integration**?

This occurs when the graphs of the 2 equations intersect. So setting them equal:

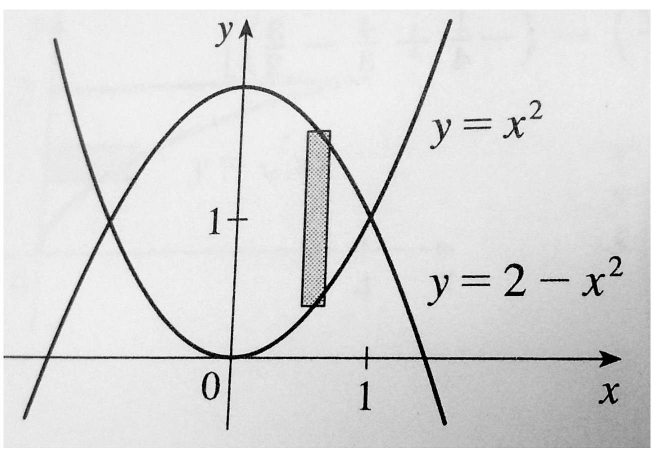
$0 = 4y^2 - y^3 \quad \Rightarrow 0 = y^2(4 - y) \quad \Rightarrow \quad$ Bounds: $0, 4$.

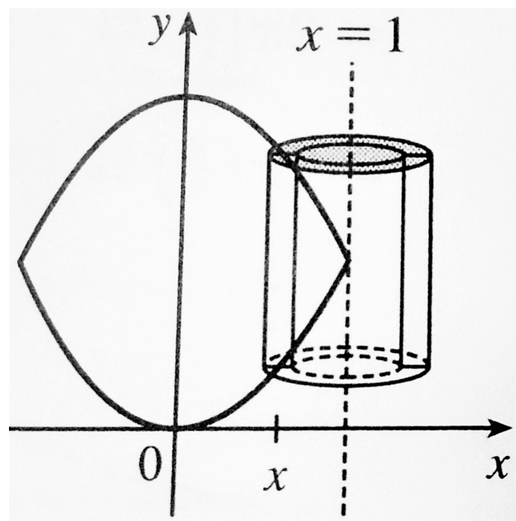
So, $V = \int_a^b [circumference] \cdot [height] \cdot [thickness] = \int_a^b (2\pi r)h(y)dy = 2\pi \int_0^4 [y(4y^2 - y^3)]dy$

$= 2\pi \int_0^4 (4y^3 - y^4)dy = 2\pi[y^4 - \frac{1}{5}y^5]_0^4 = 2\pi(256 - \frac{1024}{5}) = 2\pi(\frac{256}{5})$
$= \frac{512}{5}\pi \approx 321.70$ cubic units.

Problem 18. **Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = x^2$ and $y = 2 - x^2$ about $x = 1$.**

You should have a good idea of what the two curves look like, and observe for the intersection that $x^2 = 2 - x^2$ when $2(x^2 - 1) = 0$ or $x = \pm 1$. So we have the graphs:





What is the radius? What is the circumference?

What is the height of the cylindrical shell? Limits of Integration?

Recall that we calculated $x = \pm 1$ for the intersections, and therefore these are our limits of integration.

Radius: since the distance between the y-axis and $x = 1$ is 1 (duh), and the distance between the y-axis and x is x (double duh), the distance between $x = 1$ and x, (the radius), is $r = 1 - x$ (notice we chose this instead of $x - 1$ to ensure that the distances are positive in the interval in which we are working.

Circumference is: $2\pi r = 2\pi(1 - x)$.

Height: $(2 - x^2) - x^2 = 2 - 2x^2$.

$V = 2\pi \int_a^b r(x)h(x)dx$

$$= 2\pi \int_{-1}^{1} (1-x)(2-2x^2)dx = 2\pi(2)\int_{-1}^{1}(1-x)(1-x^2)dx = 4\pi \int_{-1}^{1}(1-x-x^2+x^3)dx$$

$$= 4\pi[x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4]_{-1}^{1}$$

$$= 4\pi\left(1 - \tfrac{1}{2}(1)^2 - \tfrac{1}{3}(1)^3 + \tfrac{1}{4}(1)^4\right) - 4\pi\left((-1) - \tfrac{1}{2}(-1)^2 - \tfrac{1}{3}(-1)^3 + \tfrac{1}{4}(-1)^4\right)$$

$$= 4\pi(1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4}) - 4\pi(-1 - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) = 4\pi(\tfrac{5}{12}) - 4\pi(-\tfrac{11}{12}) = \tfrac{5\pi}{3} + \tfrac{11\pi}{3}$$

$= \frac{16\pi}{3}$ cubic units.

ALTERNATIVELY, you can do a change of variable at the beginning of the process to move the y-axis from $x = 0$ to $x = 1$.

Change of Variable: Make the change $x \to 1-x$ (everywhere you would have written x, now write $1-x$). This moves the y-axis so that the expression $x = 1$ becomes $(1-x) = 1 \quad \Rightarrow \quad x = 0$. In other words, what used to be $x = 1$, will now be $x = 0$, the new location of the y-axis. Observe that we could have also done this with $x \to x - 1$ (since $x - 1 = 1 - 1 = 0$ when $x = 1$) but observe that with $x \to 1 - x$, as x increases from our new y-axis ($x = 0$), the value of x under the original setup moves from $x = 1$ leftwards on the graph (**verify this yourself**), which is what we want to do to integrate the shape in question. We can therefore choose to integrate from 0 to 2. If we had chosen $x \to x - 1$, we could still have integrated successfully, but we would have to choose non-intuitive bounds of integration, integrating from 2 to 0, which would still take us along the same path of integration.

Observe what happens to our bounds of integration when we make the change of variable:

$$x = \pm 1 \quad \overset{\text{change of variable}}{\Rightarrow} \quad 1 - x = \pm 1 \quad \Rightarrow \quad -x = -2 \text{ and } -x = 0 \quad \Rightarrow \quad x = 0, 2.$$

And applying this same change variable to our other quantities we now have:

Radius: x.

Circumference is: $2\pi x$.

Height: $\left(2 - (1-x)^2\right) - (1-x)^2 = 4x - 2x^2$

So, $V = 2\pi \int_0^2 x(4x - 2x^2)dx = 4\pi \int_0^2 (2x^2 - x^3)dx$

$= 4\pi[\frac{2}{3}x^3 - \frac{1}{4}x^4]_0^2 = 4\pi(\frac{2}{3}(8) - \frac{1}{4}(16)) = \frac{16}{3}\pi$ cubic units.

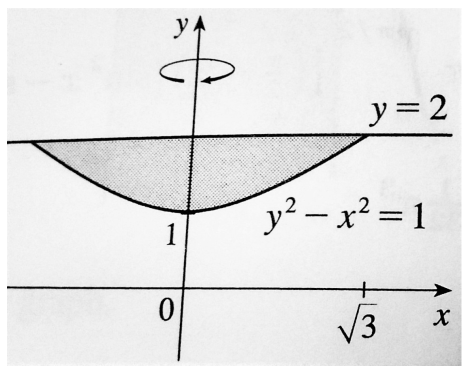
And as expected, we get the same solution.

Problem 40. **The region bounded by $y^2 - x^2 = 1$ and $y = 2$ is rotated about the y-axis. Find the volume of the resulting solid by using discs.**

To get an idea of what this looks like, note that $y = \sqrt{1+x^2}$ is kind of like the quadratic $1 + x^2$ (starting at $(0, 1)$), but the y value has been square rooted. In other words, it is a "shorter" version of this quadratic.

To find out where these intersect, note that: $(2)^2 - x^2 = 1$ when $x^2 = 3$ or $x = \pm\sqrt{3}$.

So we have:



For discs, we will be integrating along the y-axis (the axis of rotation), so we need the radius (x) of each disk in terms of y :

$y^2 - x^2 = 1 \Rightarrow x = \pm\sqrt{y^2 - 1}$.

So: $V = \pi\int_a^b (R^2 - r^2)dy$

$= \pi\int_1^2 \left[\left(\sqrt{y^2-1}\right)^2 - 0\right]dy$

$= \pi\int_1^2 (y^2 - 1)dy = \pi[\frac{1}{3}y^3 - y]_1^2$

$= \pi[(\frac{8}{3} - 2) - (\frac{1}{3} - 1)] = \frac{4}{3}\pi$ units^3.

How would we accomplish this using cylindrical shells?

What is the radius? What is the circumference?

What is the height of the cylindrical shell? Limits of Integration?

Radius: x. Circumference: $2\pi r = 2\pi x$

Height: $2 - \sqrt{x^2 + 1}$

Limits of Integration: $x = 0, \sqrt{3}$.

Therefore, $V = \int_a^b [\text{circumference}] \cdot [\text{height}] \cdot dx = \int_0^{\sqrt{3}} (2\pi x)\left(2 - \sqrt{x^2+1}\right) dx$

$$= 2\pi \int_0^{\sqrt{3}} 2x dx - 2\pi \int_0^{\sqrt{3}} x\sqrt{x^2+1}\, dx \qquad (*)$$

The first integral is easy, but for the 2nd we may need a u substitution.

Let: $u = x^2 + 1 \qquad du = 2xdx$.

So: $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\, dx = \int_0^{\sqrt{3}} x\sqrt{u}\, \frac{du}{2x} = \frac{1}{2}\int_0^{\sqrt{3}} u^{\frac{1}{2}} dy$.

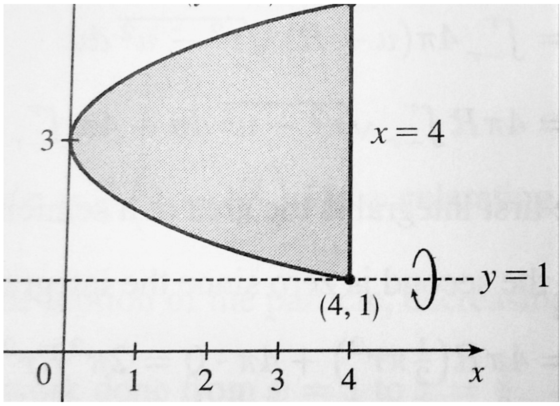
So the expression $(*)$ becomes:

$2\pi[x^2]_0^{\sqrt{3}} - \pi\int_0^{\sqrt{3}} u^{\frac{1}{2}} dy \qquad = 2\pi(3) - \pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\sqrt{3}} \qquad = 6\pi - \frac{2}{3}\pi\left[(x^2+1)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$

$$= 6\pi - \pi\left(\frac{2}{3}4^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}\right) = 6\pi - \pi\left(\frac{16}{3} - \frac{2}{3}\right) = \frac{4}{3}\pi \text{ units}^3.$$

Problem 42. **The region bounded by** $x = (y-3)^2$ **and** $x = 4$ **is rotated about** $y = 1$.

Find the volume of the resulting solid using shells.



Radius? Circumference? Height? Limits of Integration?

Radius: $y - 1$

Circumference: $2\pi(y-1)$

Height: $4-(y-3)^2$

Limits of Integration: $4 = (y-3)^2 = y^2 - 6y + 9 \quad \Rightarrow \quad y^2 - 6y + 5$

$= (y-1)(y-5) \quad \Rightarrow \quad y = 1, 5.$

$$V = \int_1^5 2\pi(y-1)\left[4-(y-3)^2\right]dy = 2\pi\int_1^5 (y-1)(-y^2+6y-5)dy = 2\pi\int_1^5(-y^3+7y^2-11y+5)dy$$

$$= 2\pi\left[-\tfrac{1}{4}y^4 + \tfrac{7}{3}y^3 - \tfrac{11}{2}y^2 + 5y\right]_1^5 = 2\pi\left(\tfrac{275}{12} - \tfrac{19}{12}\right) = \tfrac{128}{3}\pi \text{ cubic units.}$$