MATH 1271: Calculus I

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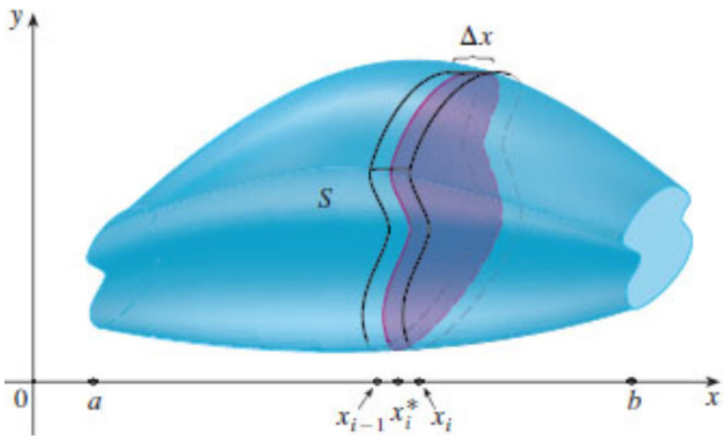
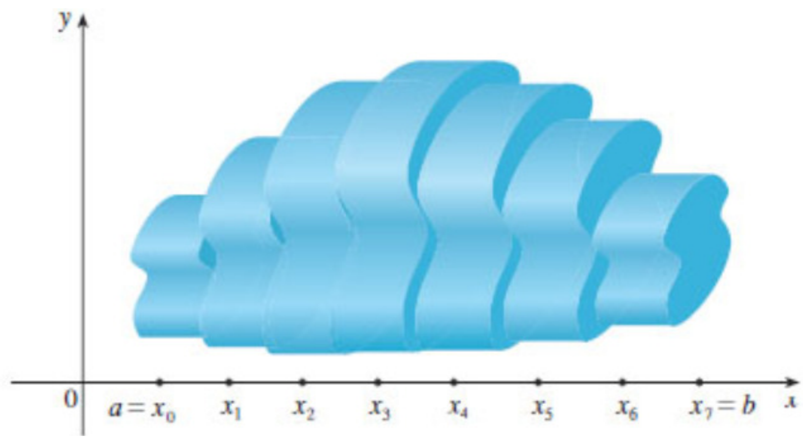
6.2 - Volumes

Review:

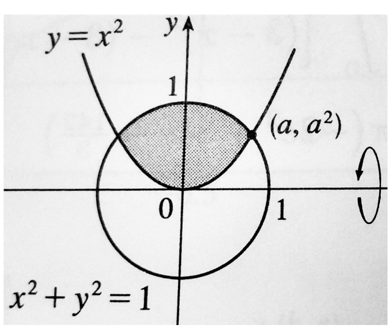




Volume: Consider a solid object in three-dimensional space situated above the positive x-axis (starting at $x = a$, and ending at $x = b$). Then consider the cross-sections you would get by cutting the object with a plane at $x = x_i^*$ where $a \leq x_i^* \leq b$. The 2D area of each cross-section can be notated $A(x_i^*)$, giving us the area function $A(x)$. In this way, divide the width of the object into n subintervals, each with width $\Delta x = \frac{b-a}{n}$. Using the Riemann Sum concept, we see that the **exact** volume is: $V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx$, where x_i^* are arbitrary points in the n intervals (left endpoints, right, midpoints, random, whatever!).

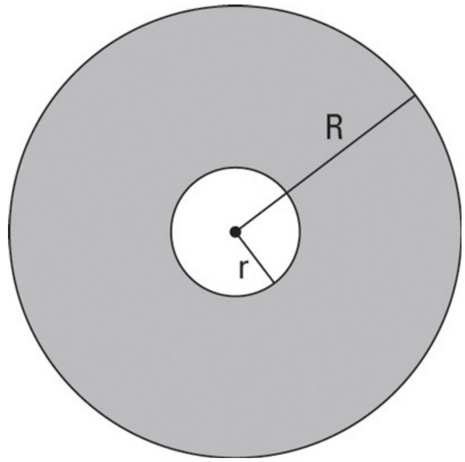
Problem 34. Set up an integral for the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x^2 + y^2 = 1$, and $y \geq 0$ about the specified line.

a. **About the x-axis.**



resulting object

Upon "sweeping out" the 3D object by rotating the region about the x-axis, we then need to determine the area of each "slice" of that 3D object. We call the "outer curve" the curve swept out by the farthest function, in this case, the circle. And we call the "inner curve" the curve swept out by the innermost function, in this case, the parabola.



$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

For each slice of the shape, we must subtract the circular area generated by the outer curve from the circular area of the inner curve. For this, we will need to know the radius of the outer curve $(x^2 + y^2 = 1)$. The radius, is just the y-value of the function. For the purposes of computing (positive) area, we use the radius of the circle, $y = \sqrt{1 - x^2}$. The radius for our inner curve is given to us as $y = x^2$.

Area of outer circle: $\pi R^2 = \pi\left(\sqrt{1 - x^2}\right)^2 = \pi(1 - x^2)$.

Area of inner circle: $\pi r^2 = \pi(x^2)^2 = \pi x^4$.

To find our bounds of integration, let’s calculate where the curves intersect.

> **Sidenote**: Normally, when we have $y = f(x)$ and $y = g(x)$, to find the points of intersection, we would set the two curves equal to each other ($f(x) = g(x)$), and then solve for x to find out the location along the x-axis where the curves’ y-values are equal to each other (they intersect). In this case, because one of our curves is defined implicitly ($x^2 + y^2 = 1$), we instead "set the two curves equal to each other" by taking the implicitly defined curve, and everywhere we see a "y," we substitute in the value of the other curve ($y = x^2$). Then, we once again solve for x to find out where along the x-axis any intersections occur.

Since $x^2 + y^2 = 1$ and $y = x^2, \ldots$

$$x^2 + (x^2)^2 = 1 \qquad \Rightarrow \qquad x^4 + x^2 - 1 = 0$$

(Note that this is quadratic in x^2, so we can use the quadratic formula!)

$$\Rightarrow x^2 = \frac{-1+\sqrt{1-4(1)(-1)}}{2(1)} = \frac{-1+\sqrt{5}}{2}$$

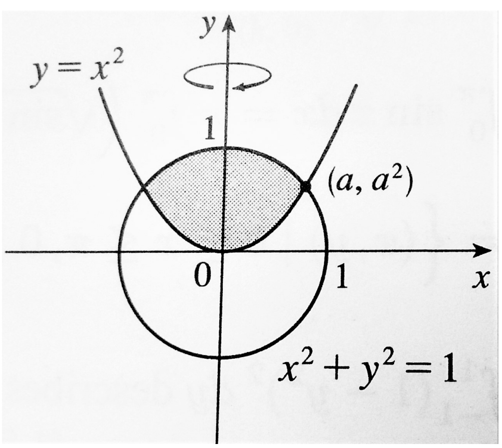
$\Rightarrow x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}} \approx \pm 0.78615$. So these are our boundary points: $\pm a$.

$$V = \int_{-a}^{a} \pi[R^2 - r^2]dx = \int_{-a}^{a} \pi[(1-x^2) - x^4]dx$$

$$= 2\pi \int_0^a (1 - x^2 - x^4)dx = 2\pi\left[x - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^a \qquad \text{(Simplified using symmetry)}$$

$$= 2\pi\left(0.78615 - \frac{(0.78615)^3}{3} - \frac{(0.78615)^5}{5}\right) \approx 3.54459 \text{ cubic units.}$$

b. **About the y-axis**.



Resulting object

The circular cross-sections are aligned horizontally instead of vertically, so let's flip the axes!

Observe we have no "inner curves," as the resulting object is convex.

Area of the lower (outer) circles (below the intersection points):

$$\pi R^2 = \pi\left(\sqrt{y}\right)^2 = \pi y.$$

Area of upper circles:

$$\pi R^2 = \pi\left(\sqrt{1-y^2}\right)^2 = \pi(1-y^2).$$

Bounds of integration:

$0 < y < a^2$ for the first integral, and $a^2 < y < 1$ for the second.

Calculating the volume:

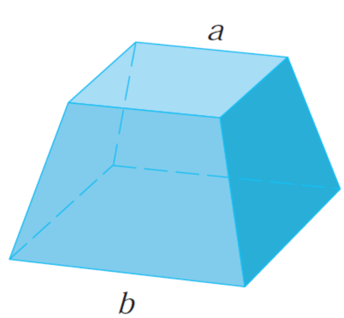
$$V = \int_0^{a^2} \pi y dy + \int_{a^2}^{1} \pi(1-y^2)dy$$

$$= \pi \int_0^{(0.78615)^2} y dy + \pi \int_{(0.78615)^2}^{1} (1-y^2)dy$$

$$= \pi\left[\frac{y^2}{2}\right]_0^{(0.78615)^2} + \pi\left[y - \frac{y^3}{3}\right]_{(0.78615)^2}^{1}$$

$$= \pi\left(\frac{(0.78615)^4}{2} - 0\right) + \pi\left(\left(1 - \frac{1^3}{3}\right) - \left((0.78615)^2 - \frac{(0.78615)^6}{3}\right)\right) \approx 0.99998 \text{ cubic units.}$$

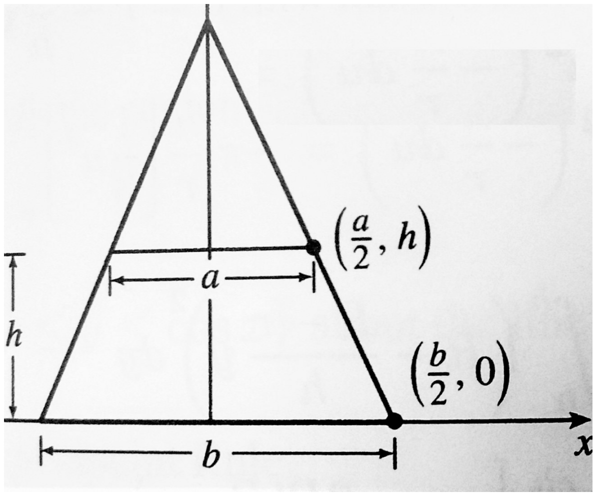
Problem 50. Using the methods of this section, find the volume of a **frustum** of a pyramid with square base of side length b, square top of side length a, and height h.



How shall we slice it?

It's easy to find the area of squares, so let us slice it horizontally, and integrate along the y-axis.

But what are the functions which will define the area?



To determine the area of each square, we will need to know the equation defining the width of the object at each point along the y-axis. To determine this width, we must know the equation of the (vertically oriented) line defining the outer wall.

Recall the equation for a (vertically oriented) line is $x = \frac{\Delta x}{\Delta y} y + x$-intercept.

So, $x = \frac{\frac{a}{2} - \frac{b}{2}}{h-0} y + \frac{b}{2} = \frac{a-b}{2h} y + \frac{b}{2}$.

But this is only half of the width of the object, so we must double it.
Then, we square the result to find the area of each square: $(2x)^2 = \left(\frac{a-b}{h} y + b\right)^2$.

Now we integrate that area along the height to find the volume:

$V = \int_a^b A(y)dy = \qquad ?$

$$= \int_0^h \left(\frac{a-b}{h} y + b\right)^2 dy = \int_0^h \left(\frac{(a-b)^2}{h^2} y^2 + 2\frac{b(a-b)}{h} y + b^2\right) dy$$

$$= \left[\frac{(a-b)^2}{3h^2} y^3 + \frac{b(a-b)}{h} y^2 + b^2 y\right]_0^h = \frac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h$$

$$= \frac{1}{3}(a^2 - 2ab + b^2 + 3ab)h = \frac{1}{3}(a^2 + ab + b^2)h \text{ cubic units.}$$

What happens if $a = b$? **What happens if $a = 0$?**

In the graph, when $a = b$, we see we have a **rectangular** solid, and our equation above becomes the usual volume formula of b^2h cubic units!

In the graph, when $a = 0$, we have a square **pyramid**, and our equation above becomes the usual pyramid volume formula of $\frac{1}{3}b^2h$ cubic units!

Problem 54. Find the volume of a shape with a circular base of radius r, whose parallel cross-sections, being perpendicular to the base, are squares.

Each slice will therefore be a square, whose side length is determined by width of the circle at that point along x.

$x^2 + y^2 = r^2$, therefore $y = \pm\sqrt{r^2 - x^2}$, and the width of the circle is $2\sqrt{r^2 - x^2}$.

So, $A(x) = \left(2\sqrt{r^2 - x^2}\right)^2 = 4(r^2 - x^2)$.

$V = \int_{-r}^{r} A(x)dx$

$= 2\int_{0}^{r} 4(r^2 - x^2)dx$

$= 8[r^2x - \frac{1}{3}x^3]_0^r = 8(r^3 - \frac{1}{3}r^3) = 8(\frac{2}{3}r^3) = \frac{16}{3}r^3$ cubic units.