MATH 1271: Calculus I

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5.4 - Indefinite Integrals and the Net Change Theorem

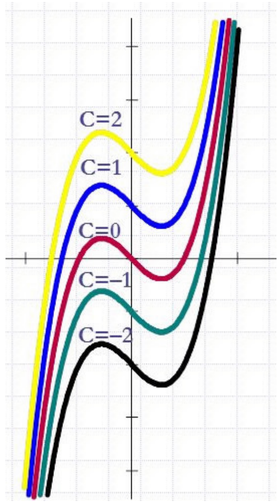
Review

Indefinite Integrals: $\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

For example $\int x^3 dx = \frac{x^4}{4} + C$, because $\frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$ (for any C !)

Therefore, an indefinite integral $\int f$ represents a **family of** *functions*,
as opposed to a definite integral $\int_a^b f$ which just represents a *constant*.

And by FTC2: $\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b = F(b) - F(a)$.



family of functions: $\int f$

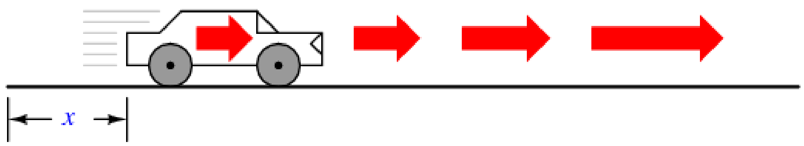
Table of Indefinite Integrals and Properties

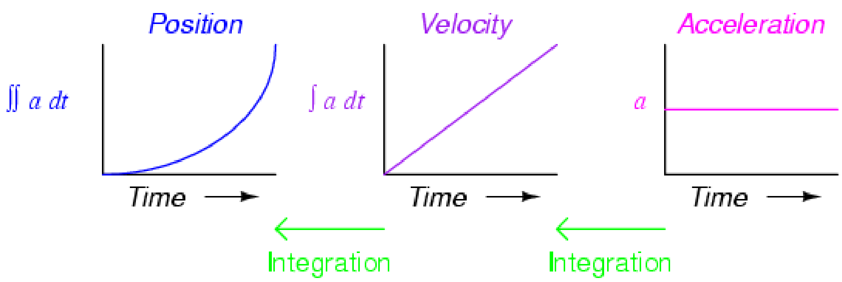
$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int adx = ax + C$, where a is some constant	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int e^x dx = e^x + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (when $n \neq -1$) (backwards power rule)
$\int x^{-1}dx = \int \frac{1}{x}dx = \ln\lvert x\rvert + C$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \sec^2 x dx = \tan x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \frac{1}{x^2+1}dx = \tan^{-1} x + C$

Net Change Theorem

The integral of a rate of change (of some quantity) is the net change (of that quantity) over the interval defined by the bounds of integration. So, $\int_a^b F'(x)dx = F(b) - F(a)$ represents the net change of F over the interval $[a,b]$.

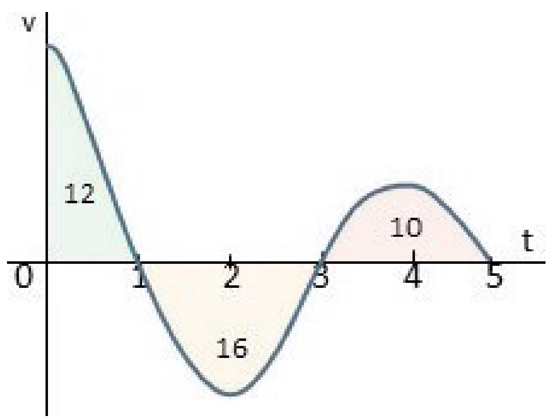
Examples





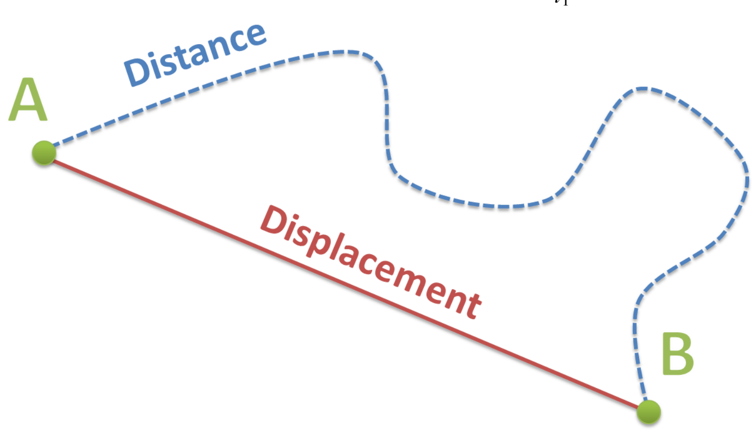
Position $x(t)$ and Velocity $v(t)$ functions:

Displacement of a car's position between t_1 and t_2 (distance from starting position) is calculated by integrating the velocity: $\int_{t_1}^{t_2} v(t)dt = x(t_2) - x(t_1)$



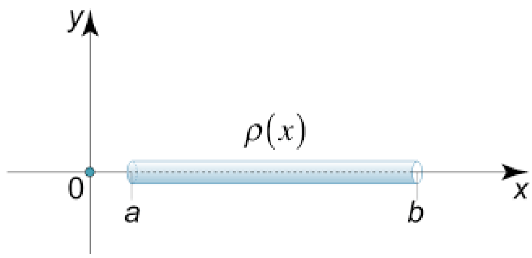
Net change is $12 - 16 + 10 = 6$.

Distance traveled (as recorded by the odometer) between t_1 and t_2 is calculated by integrating the absolute value of the velocity: $\int_{t_1}^{t_2} |v(t)|dt$



Distance traveled

Mass of a Rod:
Notate the total mass of a rod between 0 and x as $m(x)$.
The density can then be calculated as $\rho(x) := m'(x)$
($m'(x)$ is how quickly the total mass is changing as you move along x).
Net change of mass between a and b is then: $\int_a^b \rho(x)dx = m(b) - m(a)$.



Problem 44 Evaluate the integral: $\int_0^2 |2x-1|dx$.

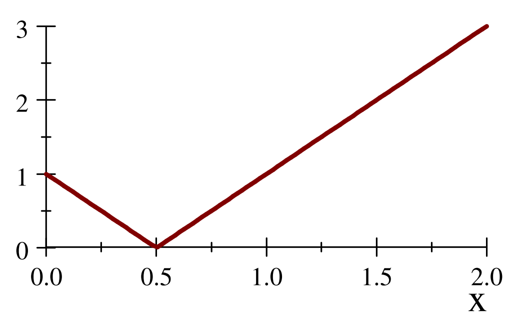
$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0, \\ -(2x-1) & \text{if } 2x-1 < 0, \end{cases}$$

$$= \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2}, \\ 1-2x & \text{if } x < \frac{1}{2}. \end{cases}$$

Therefore, $\int_0^2 |2x-1|dx = \int_0^{\frac{1}{2}} (1-2x)dx + \int_{\frac{1}{2}}^2 (2x-1)dx$

$$= [x-x^2]_0^{\frac{1}{2}} + [x^2-x]_{\frac{1}{2}}^2$$

$$= [(\tfrac{1}{2}-\tfrac{1}{4})-0] + [(4-2)-(\tfrac{1}{4}-\tfrac{1}{2})] = \tfrac{1}{4}+2-(-\tfrac{1}{4}) = \tfrac{5}{2}.$$



$|2x-1|$

Problem 54 A honeybee population $n(t)$ starts with $n(0) = 100$ bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)dt$ represent?

By the Net Change Theorem,

$$\int_0^{15} n'(t)dt = n(15) - n(0) = n(15) - 100,$$

represents the increase in the bee population over 15 weeks.

So, $100 + \int_0^{15} n'(t)dt = 100 + (n(15) - 100) = n(15),$

represents the total bee population after 15 weeks.

Problem 60. For a particle moving along a line, the velocity function (in meters per second) is $v(t) = t^2 - 2t - 8$. Find $(a.)$ the displacement and $(b.)$ the distance traveled by the particle during the time interval $1 \leq t \leq 6$.

a. Displacement:

$$\int_1^6 (t^2 - 2t - 8)dt$$

$$= [\tfrac{1}{3}t^3 - t^2 - 8t]_1^6$$

$$= (72 - 36 - 48) - (\tfrac{1}{3} - 1 - 8) = -\tfrac{10}{3}m.$$

b. Distance traveled:

Observe that: $t^2 - 2t - 8 = (t-4)(t+2)$.

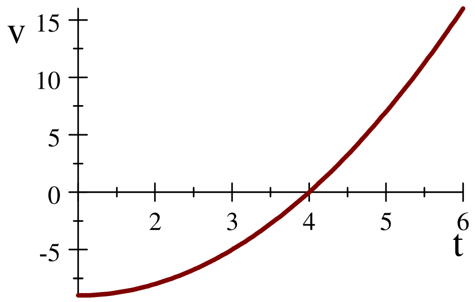
So it is negative on $1 \leq t < 4$, and nonnegative on $4 \leq t < 6$.

Therefore, we want to calculate $\int_1^6 |t^2 - 2t - 8|dt$

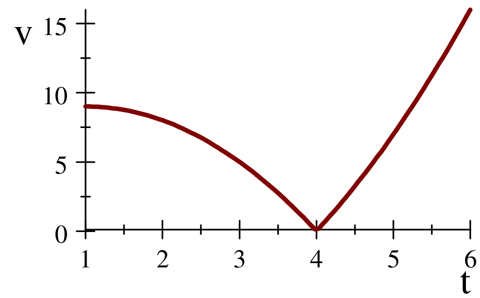
$$= \int_1^4 (-t^2 + 2t + 8)dt + \int_4^6 (t^2 - 2t - 8)dt$$

$$= [-\tfrac{1}{3}t^3 + t^2 + 8t]_1^4 + [\tfrac{1}{3}t^3 - t^2 - 8t]_4^6$$

$$= (-\tfrac{64}{3} + 16 + 32) - (-\tfrac{1}{3} + 1 + 8) + (72 - 36 - 48) - (\tfrac{64}{3} - 16 - 32) = \tfrac{98}{3}m.$$



$t^2 - 2t - 8$



$|t^2 - 2t - 8|$

Problem 62 The acceleration function $a(t) = 2t + 3$ (in m/s^2) and the initial velocity $v(0) = -4$ are defined for a particle moving along a line. Find (i) the velocity at time t and (ii) the distance traveled during the time interval $0 \le t \le 3$.

i. $v'(t) = a(t) = 2t + 3$

$$v(t) = \int a(t) = t^2 + 3t + C$$

Noting the initial condition, we can solve for C : $v(0) = C = -4$.

Therefore, $v(t) = t^2 + 3t - 4$.

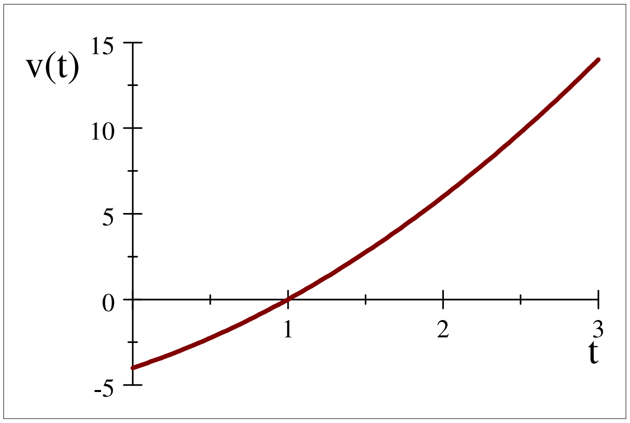
ii. Distance traveled: Since we have $v(t) = t^2 + 3t - 4$ over $0 \le t \le 3$, ...

$$\int_0^3 |t^2 + 3t - 4| dt = \int_0^3 |(t+4)(t-1)| dt$$

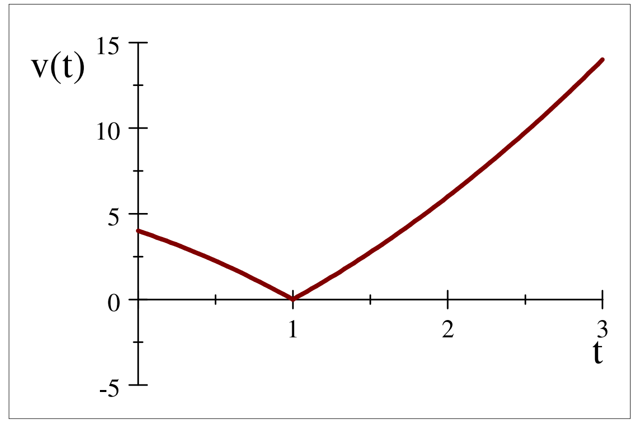
$$= \int_0^1 (-t^2 - 3t + 4) dt + \int_1^3 (t^2 + 3t - 4) dt$$

$$= [-\tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 4t]_0^1 + [\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t]_1^3$$

$= (-\frac{1}{3} - \frac{3}{2} + 4) + (9 + \frac{27}{2} - 12) - (\frac{1}{3} + \frac{3}{2} - 4) = \frac{89}{6} \approx 14.8$ meters.



Graph of $t^2 + 3t - 4$



Graph of $|t^2 + 3t - 4|$