# MATH 1271: Calculus I

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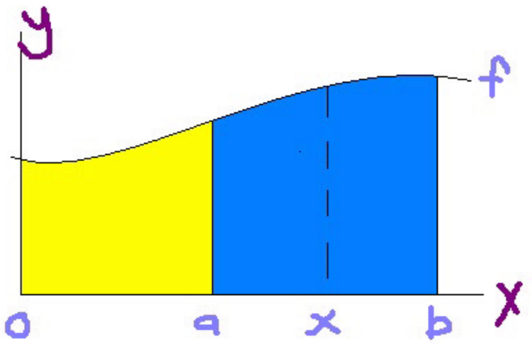
## 5.3 - Fundamental Theorem of Calculus

### Review

Recall that $\int_a^b f(t)dt$ is a constant representing the area under the curve of $f$ over the interval $[a,b]$. Now, let $g(x) = \int_a^x f(t)dt$. Notice the variable $x$ in the upper bound of integration, and that we've just created a **function** $g(x)$ that represents the area under the curve of $f$ over the interval $[a,x]$, for any given $x$.




Now suppose $f$ is continuous on $[a,b]$, with $a \le x \le b$ :
**Fundamental Theorem of Calculus** (**FTC**)

- **FTC1**: $g(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$, differentiable on $(a,b)$, and $g'(x) = f(x)$.
- **FTC2**: $\int_a^b f(x)dx = F(b) - F(a)$, where $F$ is ANY antiderivative of $f$ (i.e., $F' = f$).

**Alternate Formulation of FTC**:

- **FTC1**: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.
- **FTC2**: $\int_a^b F'(x)dx = F(b) - F(a)$.

**Notation you'll need**: $\int_a^b F'(x)dx = [F(x)]_a^b := F(x)|_a^b := F(b) - F(a)$.

More Generally:
$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$, because of the chain rule.
And even more generally: $\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = \frac{d}{dx}\left(\int_{h(x)}^0 f(t)dt + \int_0^{g(x)} f(t)dt\right)$

$$= \frac{d}{dx}\left(-\int_0^{h(x)} f(t)dt + \int_0^{g(x)} f(t)dt\right) = -\frac{d}{dx}\int_0^{h(x)} f(t)dt + \frac{d}{dx}\int_0^{g(x)} f(t)dt$$
$$= -f(h(x)) \cdot h'(x) + f(g(x)) \cdot g'(x).$$

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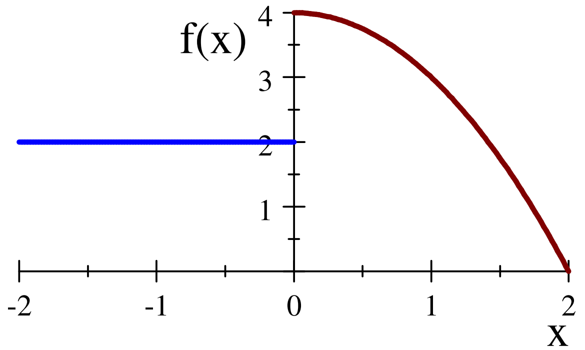
**Problem** 44. **Evaluate the integral**: $\int_{-2}^{2} f(x)dx$, where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0, \\ 4 - x^2 & \text{if } 0 < x \le 2. \end{cases}$




$$\int_{-2}^{2} f(x)dx = \int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx$$

$$= \int_{-2}^{0} 2dx + \int_{0}^{2} (4 - x^2)dx$$

$$= [2x]_{-2}^{0} + [4x - \tfrac{1}{3}x^3]_{0}^{2}$$

Recall from the review that: $[2x]_{-2}^{0} = 2 \cdot 0 - 2 \cdot (-2) = 4.$

Also: $[4x - \frac{1}{3}x^3]_0^2 = \left(4(2) - \frac{1}{3}(2)^3\right) - \left(4(0) - \frac{1}{3}(0)^3\right) = \frac{16}{3}.$

So: $\int_{-2}^{2} f(x)dx = 4 + \frac{16}{3} = \frac{28}{3}.$

Note that $f$ is integrable by Theorem 3 in section 5.2 because the function only has a finite number of jump discontinuities (just one) in the range of integration.

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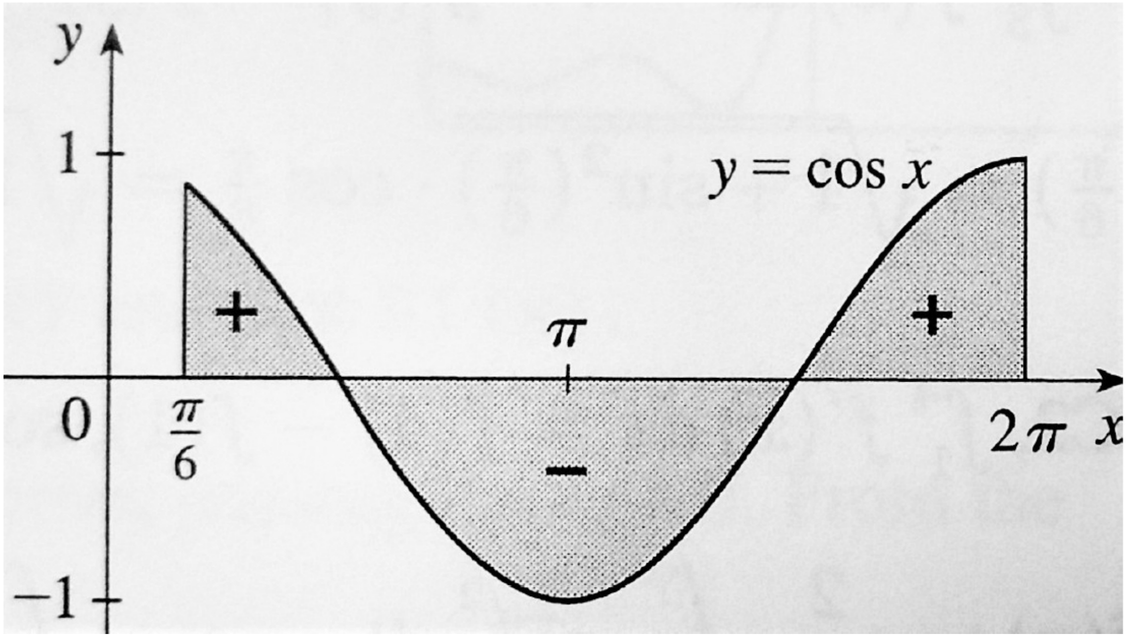
**Problem** 54. Evaluate $\int_{\frac{\pi}{6}}^{2\pi} \cos x dx$ and interpret it as a difference of areas. Illustrate with a sketch.

$$\int_{\frac{\pi}{6}}^{2\pi} \cos x dx = [\sin x]_{\frac{\pi}{6}}^{2\pi}$$

$$= \sin(2\pi) - \sin(\tfrac{\pi}{6})$$

$= 0 - \frac{1}{2} = -\frac{1}{2}.$

Observe that $\cos x$ is negative between $\frac{\pi}{2}$ and $\frac{3}{2}\pi$. But elsewhere on the interval, it is positive. Therefore, our integral represents the net area under the curve for these positive and negative values of $\cos x$

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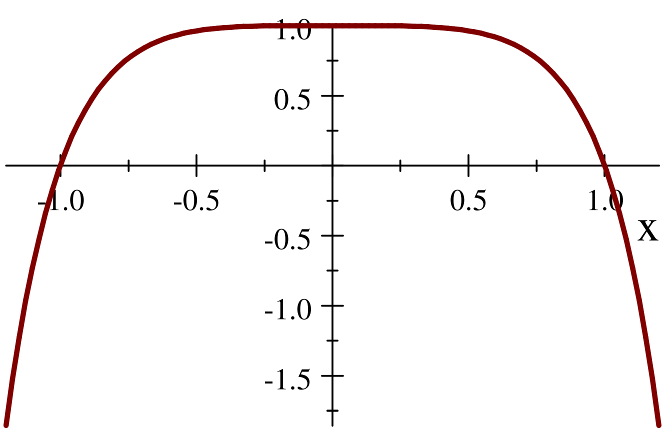
**Problem** 60. If $f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$, on what interval is the function increasing?

$f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$ is increasing when $f'(x) = (1-x^2)e^{x^2}$ is positive.

Since $e^{x^2} > 0$, $f'(x) > 0$ requires that $1 - x^2 > 0$,

resulting in: $|x| < 1$, so $f$ is increasing on $(-1, 1)$.




$(1-x^2)e^{x^2}$

**Problem** 62. If $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$ and $g(y) = \int_3^y f(x)dx$, find $g''(\frac{\pi}{6})$.

$g''(\frac{\pi}{6})$ means the 2nd derivative evaluated at $\frac{\pi}{6}$. So let's first take our derivatives…

$$g'(y) = f(y) = \int_0^{\sin y} \sqrt{1+t^2}\,dt.$$

$$g''(y) = \frac{dg'(y)}{dy} = f'(y) = \frac{d}{dy}\left[\int_0^{\sin y} \sqrt{1+t^2}\,dt\right]$$

Let: $u = \sin y$ and $\frac{du}{dy} = \cos y$.

Using the chain rule: $\frac{d}{dy} = \frac{d}{du}\frac{du}{dy}$.

So, $g''(y) = \frac{d}{du}\left[\int_0^{u} \sqrt{1+t^2}\,dt\right]\frac{du}{dy}$

$$= \sqrt{1+u^2} \bullet \frac{du}{dy} = \sqrt{1+\sin^2 y} \bullet \cos y.$$

So, $g''(\frac{\pi}{6}) = \sqrt{1+\sin^2(\frac{\pi}{6})} \bullet \cos\frac{\pi}{6} = \sqrt{1+(\frac{1}{2})^2} \bullet \frac{\sqrt{3}}{2} = \frac{\sqrt{5}}{2} \bullet \frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$.

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**Problem** 69. If $f(1) = 12$, $f'$ is continuous, and $\int_1^4 f'(x)dx = 17$,
then what is the value of $f(4)$?

By FTC2, $\int_1^4 f'(x)dx = f(4) - f(1)$,

so $17 = f(4) - 12 \quad \Rightarrow \quad f(4) = 17 + 12 = 29$.

**Problem** 60. Find the derivative of the function: $y = \int_{1-2x}^{1+2x} t \sin t dt$

$$y = \int_{1-2x}^{0} t \sin t dt + \int_{0}^{1+2x} t \sin t dt$$

$$\frac{d}{dx} y = -\frac{d}{dx} \int_{0}^{1-2x} t \sin t dt + \frac{d}{dx} \int_{0}^{1+2x} t \sin t dt$$

$$= -\frac{d(1-2x)}{dx} \frac{d}{d(1-2x)} \int_{0}^{1-2x} t \sin t dt + \frac{d(1+2x)}{dx} \frac{d}{d(1+2x)} \int_{0}^{1+2x} t \sin t dt \qquad \left[\text{chain rule! } \frac{d}{dx} = \frac{d(1-2x)}{dx} \frac{d}{d(1-2x)}\right]$$

$$= 2 \frac{d}{d(1-2x)} \int_{0}^{1-2x} t \sin t dt + 2 \frac{d}{d(1+2x)} \int_{0}^{1+2x} t \sin t dt$$

$$= 2(1-2x) \sin(1-2x) + 2(1+2x) \sin(1+2x).$$