MATH 1271: Calculus I

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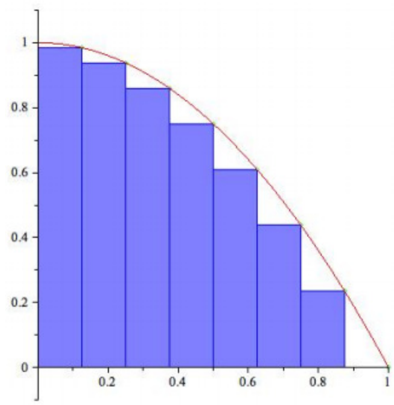
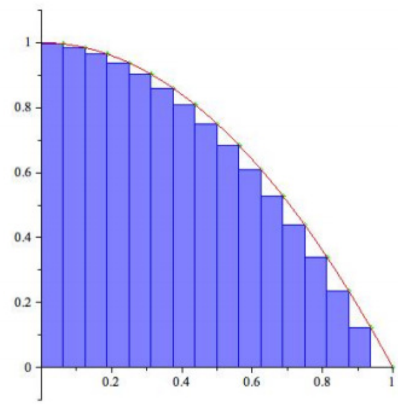
5.1 - Riemann Sums and Distances

Review:

Area Under the Curve:



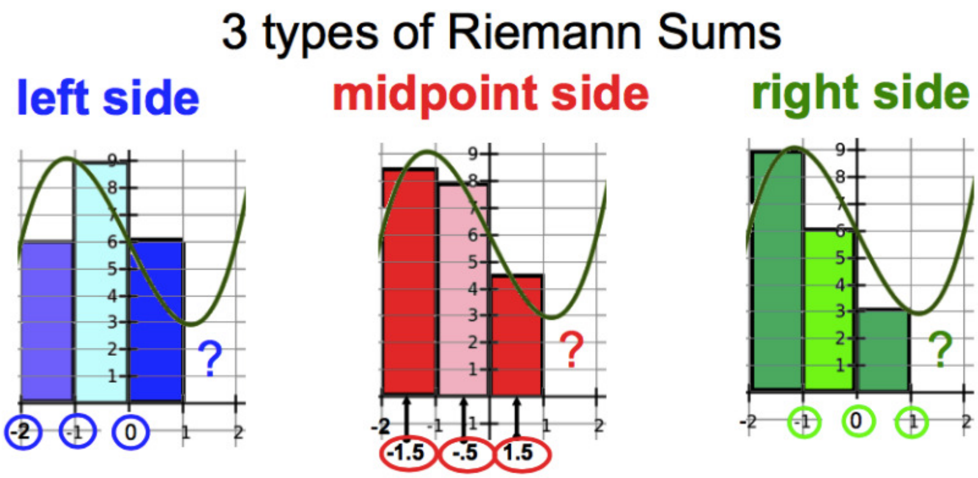


Let's say you are given a continuous function f, and you want to find the area under the curve from $x = a$ to $x = b$. We'll do it by adding up the areas of some rectangles. First, break up the interval $[a,b]$ into some number n of subintervals (rectangles) with equal length. So, each subinterval has length $\Delta x = \frac{b-a}{n}$. Let $\{x_1, x_2, \ldots, x_n\}$ be the labels for the **right** endpoints of these intervals, so $x_i = a + i\Delta x$, where i goes from 1 to n. So the area of each rectangle would be the width Δx times the height ($f(x_1)$). So, an estimate of the area under the curve would be given by $A \approx$ (area of 1st rectangle)+(area of 2nd rectangle)+$\ldots$+(area of n^{th} rectangle) $= f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$. But if you take an increasingly larger number of (and therefore shorter) intervals, the estimates would become more accurate such that the **exact** area under the curve can given by:
$A = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x] := \lim_{n\to\infty} R_n$.

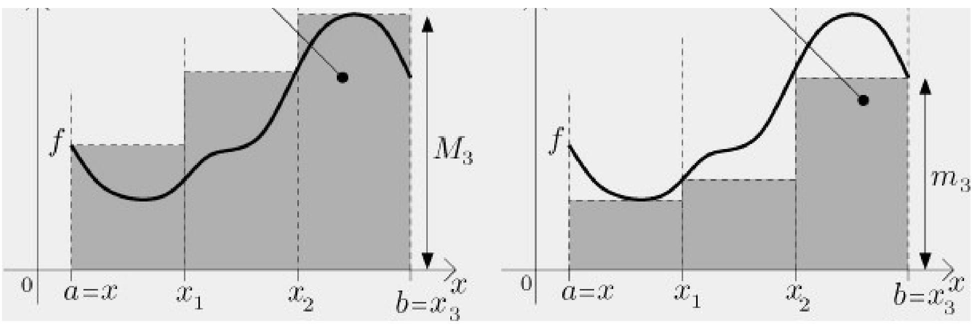
For example, if we had the interval $[0,2]$, and chose $n = 4$, the right endpoints would be $\{x_1, x_2, x_3, x_4\} = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ with $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

Note we could have easily defined the area with the **left** endpoints $\{x_0, x_1, \ldots, x_{n-1}\}$ by $A = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x] := \lim_{n\to\infty} L_n$. And our example would then have been $\{x_0, x_1, x_2, x_3\} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}\right\}$.

We can also use midpoints $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$.
Alternatively, we can use arbitrary sample points $\{x_1^*, x_2^*, \ldots, x_n^*\}$, for each subinterval (located in a different place in each interval!). Interestingly, each of these can be used to determine the same area, in the limit.



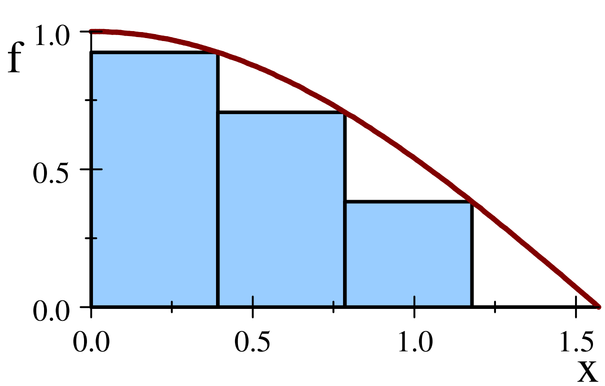
In particular, if you choose the sample points such that you maximize the area of each rectangular column, the resulting sum is called the **upper sum**. Similarly, one can choose points to find the **lower sum**.



Upper and Lower sum

Note: If a function is **increasing**, using **left endpoints** will give you the **lower sum**, and **right endpoints** the **upper sum**. Relatedly, a decreasing function will do the opposite.

Problem 3(a). Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ using four approximating rectangles and right endpoints. Is this an over or an under-estimate?



$n = 4.$

$$\Delta x = \frac{\text{Total Distance}}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}.$$

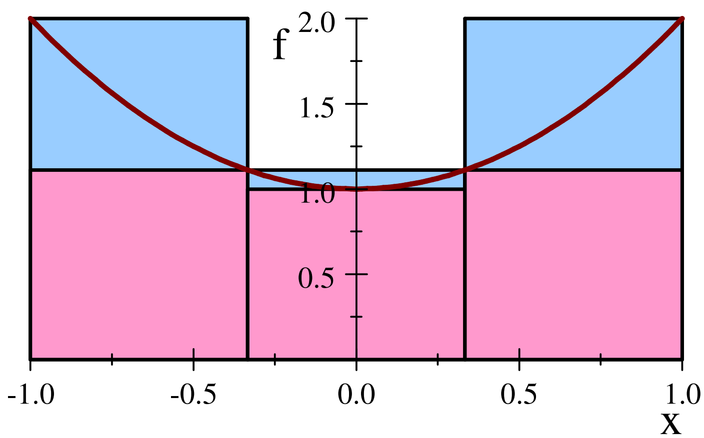
$$R_4 \approx [\Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4)] = [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\Delta x$$

$$= (\cos\tfrac{\pi}{8} + \cos\tfrac{2\pi}{8} + \cos\tfrac{3\pi}{8} + \cos\tfrac{\pi}{2})\tfrac{\pi}{8} \approx 0.791.$$

Over or under-estimate?

(b) Repeat part a using left endpoints. (Take Home Exercise!)

Problem 8. Evaluate the upper and lower sums for $f(x) = 1 + x^2$, $-1 \le x \le 1$, with $n = 3$. Illustrate with diagram.



$\Delta x = \frac{2}{n} = \frac{2}{3}$. Let's do Upper Sum first :

$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$. Which points to choose?

$$= [f(-1) + f(\pm\tfrac{1}{3}) + f(1)](\tfrac{2}{3})$$

$$= (2 + \tfrac{10}{9} + 2)(\tfrac{2}{3}) = \tfrac{92}{27} \approx 3.41.$$

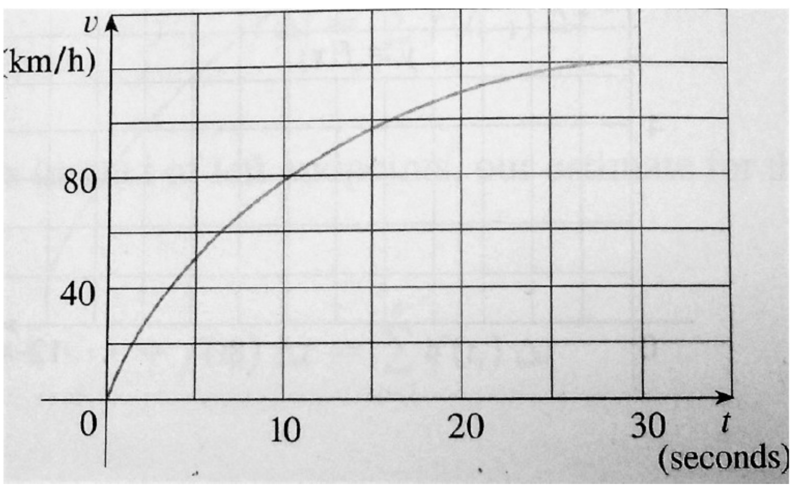
Lower Sum =

$[f(x_1)+f(x_2)+f(x_3)]\Delta x$ Which points to choose?

$= [f(-\frac{1}{3})+f(0)+f(\frac{1}{3})](\frac{2}{3})$

$= (\frac{10}{9}+1+\frac{10}{9})(\frac{2}{3}) = \frac{58}{27} \approx 2.15.$

Problem 18. The velocity graph of a car accelerating from rest to a speed of 120 *km/h* over a period of 30 seconds is shown. Estimate the distance traveled during this period.



Recall (or learn for the first time) that the distance traveled by an object is equal to the area-under-the-curve of the velocity function for that object.

For an increasing function, using left endpoints gives us an under-estimate and using right endpoints results in an over-estimate. So let us use a midpoint estimate. Observe we weren't given a mathematical function for velocity, so we will have to use the graph to estimate the height of the function at our sample points.

We will use $n = 6$.

$\Delta t = \frac{30-0}{6} = 5\,\text{sec} = \frac{5}{3600}\,hours = \frac{1}{720}\,hours.$ We switched to hours since the velocity is in hours.

Starting at 0, if our step size is $\Delta t = 5$, observe that the first midpoint is $\bar{x}_1 = 2.5$, then we add our step size to this for each subsequent sample point.

$$\text{Distance Traveled} = \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)]$$

$$\approx \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725\ km.$$

For a very rough check on the above calculation, we can draw a line from $(0,0)$ to $(30,120)$ and calculate the area of the triangle: $\frac{1}{2}\left(120\ km/hr\right)(30\,\text{sec})\left(\frac{1\,hr}{3600\ \text{sec}}\right) = 0.5\,km$. Looking at the graph, this triangle area is clearly an under-estimate, suggesting our midpoint estimate of 0.725 is reasonable.

Problem 15. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

$t(h)$	0	2	4	6	8	10
$r(t)\left(\frac{L}{h}\right)$	8.7	7.6	6.8	6.2	5.7	5.3

$\Delta t = 2$, and $n = 5$.

Upper: $[f(x_0) + f(x_1) + \ldots + f(x_n)]2 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)2 = 70.$

Lower: $[f(x_1) + f(x_2) + \ldots + f(x_{n-1})]2 = (7.6 + 6.8 + 6.2 + 5.7)2 = 52.6.$