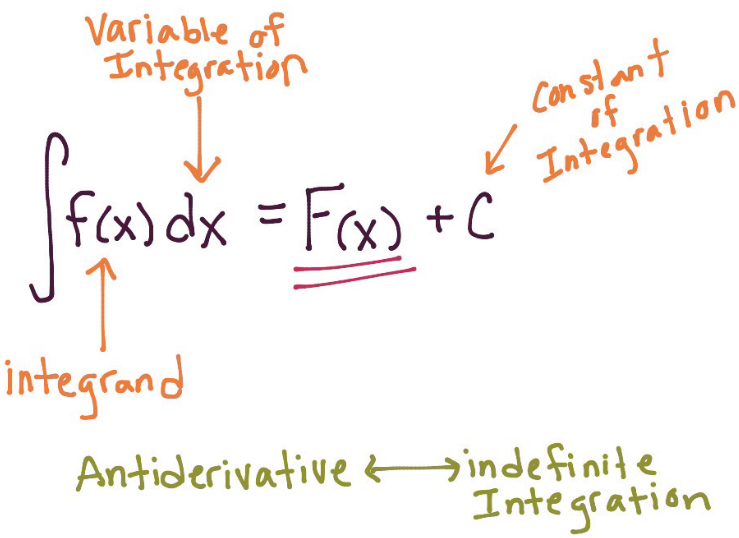
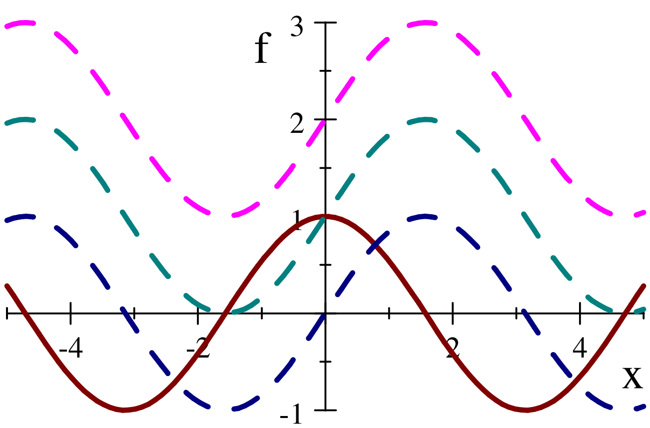
MATH 1271: Calculus I

Discussion Instructor: Jodin Morey
moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

4.9 - Anti-derivatives

Review





$f = \cos x$ (solid), $F = \sin x + C$ (dashed)

Definition: A function F is called an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Most General Anti-derivative: If F is an anti-derivative of f on an interval I, then the most general anti-derivative of f on I is $F(x) + C$, where C is an arbitrary constant.

Table of Anti-differentiation Formulas:

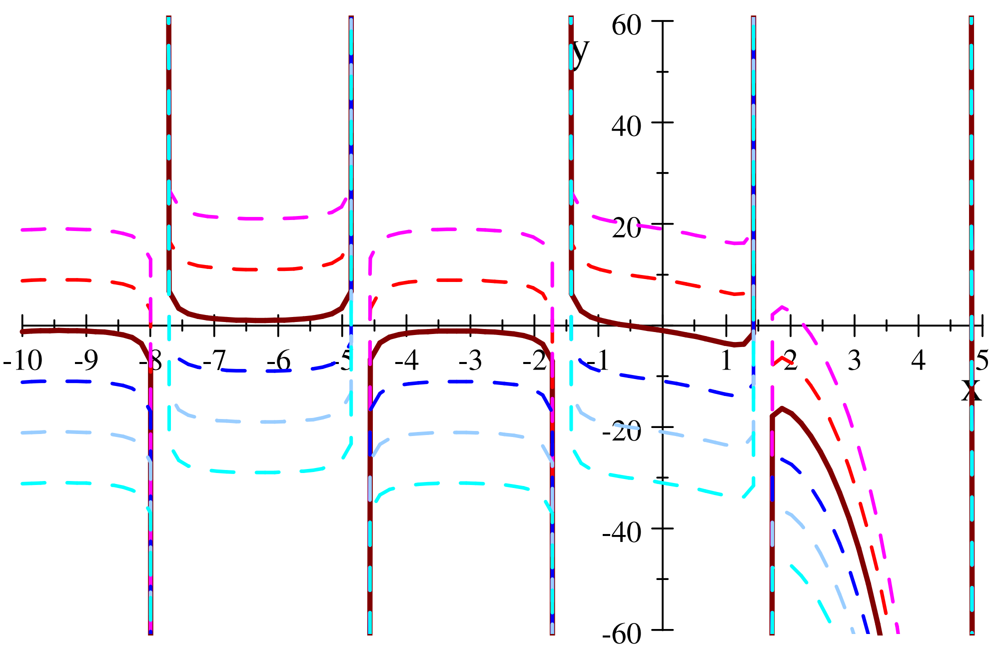
Function	Particular anti-derivative	Function	Particular anti-derivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\frac{1}{\sqrt{1-x^2}}$	$\arccos x$
b^x	$\frac{b^x}{\ln b}$	$\frac{1}{1+x^2}$	$\arctan x$

These are useful in solving **differential equations** (equations which include derivatives), for example finding $f(x)$ when given $f'(x) = 4e^x$. In this case, we see that $f(x) = 4e^x + C$ for all C is the most general anti-derivative.

Problem 16. Find the most general anti-derivative of $r(\theta) = \sec\theta\tan\theta - 2e^\theta$. (Check your answer by differentiating)

$R(\theta) = \sec\theta - 2e^\theta + C.$

Observe that this is a "family of solutions," an infinite number of functions because C can take any value.



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$\sec\theta - 2e^\theta + C, \text{ for } C \in \{-30, -20, -10, 0, 10, 20\}$

Problem 22. Find the most general anti-derivative of $f(x) = \frac{2+x^2}{1+x^2}$.

Looking for a function $F(x)$ such that $F'(x) = \frac{2+x^2}{1+x^2}$.

If you see denominators like $1 + x^2$, or $\sqrt{1 - x^2}$, then you want to think of the derivatives of inverse trigonometric functions.

In this case, $(\arctan x)' = \frac{1}{1+x^2}$.

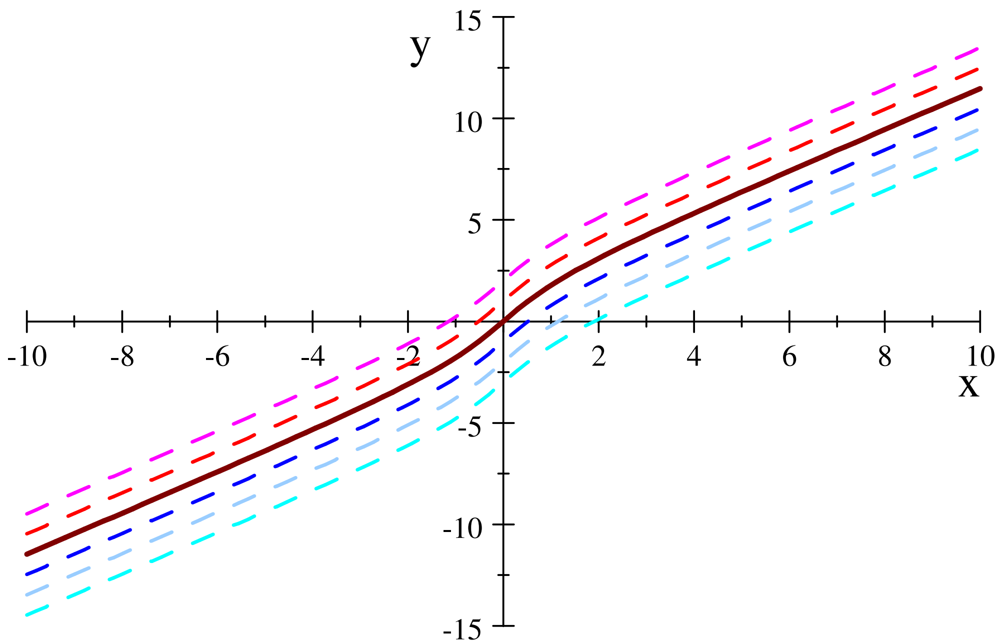
So we want to break up $f(x)$ into 2 parts, one of which $(\frac{1}{1+x^2})$ we've solved for in the previous line.

So we need $G(x)$ such that $\frac{2+x^2}{1+x^2} = G(x) + \frac{1}{1+x^2}$.

Solving for the unknown: $G(x) = \frac{2+x^2}{1+x^2} - \frac{1}{1+x^2} = \frac{1+x^2}{1+x^2} = 1$.

Therefore, $\frac{2+x^2}{1+x^2} = 1 + \frac{1}{1+x^2}$,

and $F(x) = x + \arctan x + C$.



$x + \tan^{-1} x + C$, for $C \in \{-3, -2, -1, 0, 1, 2\}$

Problem 36. Find f, when $f'(x) = \frac{x^2-1}{x}$.

Also impose the requirement that: $f(1) = \frac{1}{2}$, and $f(-1) = 0$. (these are called "initial conditions")

Often it's a good idea to simplify compound numerators as $f'(x) = \frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$.

Our initial thought may be to make $f(x) = \frac{x^2}{2} - \ln|x| + C$.

However, notice from our initial conditions that $f(1) = \frac{1^2}{2} - \ln 1 + C = \frac{1}{2}$, or $C = 0$.

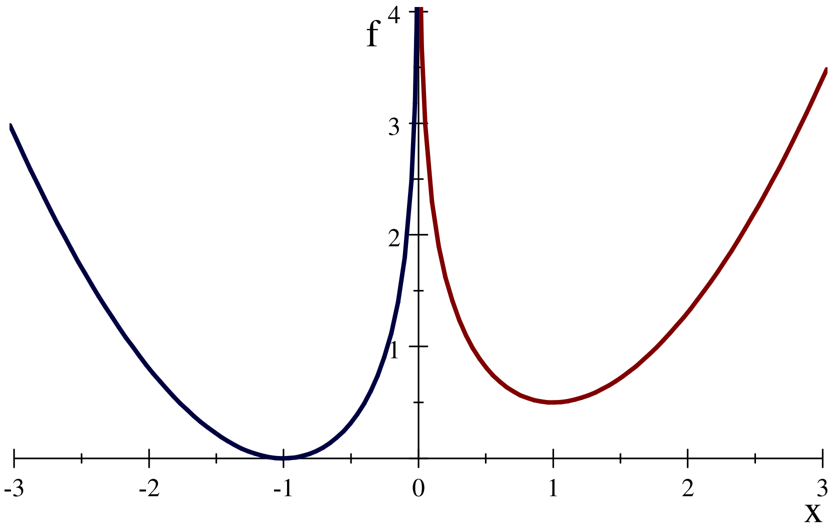
And we also have $f(-1) = \frac{(-1)^2}{2} - \ln 1 + C$, or $C = \frac{1}{2}$. Did we made a mistake?

Observe that (since $\ln 0$ isn't a thing) we have two disconnected parts of our domain $(-\infty, 0)$ and $(0, \infty)$. Therefore there is the possibility of different constants of integration on each of these intervals. So, to completely cover the possibilities, we must rewrite our anti-derivative as the piecewise function:

$$f(x) = \begin{cases} \frac{1}{2}x^2 - \ln x + C_1 & \text{if } x > 0 \\ \frac{1}{2}x^2 - \ln(-x) + C_2 & \text{if } x < 0 \end{cases}.$$

Now the previous calculations give us $C_1 = 0$, and $C_2 = \frac{1}{2}$.

$$\text{Thus, } f(x) = \begin{cases} \frac{1}{2}x^2 - \ln x & \text{if } x > 0 \\ \frac{1}{2}x^2 - \ln(-x) - \frac{1}{2} & \text{if } x < 0. \end{cases}$$



$\frac{1}{2}x^2 - \ln(-x) - \frac{1}{2}$ and $\frac{1}{2}x^2 - \ln x$